ADAPTIVE CHAOS SYNCHRONIZATION OF UNCERTAIN HYPERCHAOTIC LORENZ AND HYPERCHAOTIC LÜ SYSTEMS

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ABSTRACT

In this paper, we apply adaptive control method to derive new results for the global chaos synchronization of identical hyperchaotic Lorenz systems (Jia, 2007), identical hyperchaotic Lü systems (Chen, et al., 2006) and non-identical hyperchaotic Lorenz and hyperchaotic Lü systems. In this paper, we shall assume that the parameters of both master and slave systems are unknown and we devise adaptive synchronizing schemes using the estimates of parameters for both master and slave systems. Our adaptive synchronization schemes derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is very effective and convenient to synchronize identical and non-identical hyperchaotic Lorenz and hyperchaotic Lü systems. Numerical simulations are shown to demonstrate the effectiveness of the proposed adaptive synchronization schemes for the uncertain hyperchaotic systems addressed in this paper.

KEYWORDS


1. INTRODUCTION

Chaotic systems are dynamical systems that are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the butterfly effect [1].

Since the seminal work by Pecora and Carroll ([2], 1990), chaos synchronization problem has been studied extensively and intensively in the literature [2-29]. Chaos theory has been applied to a variety of fields such as physical systems [3], chemical systems [4], ecological systems [5], secure communications [6-8], etc.

In most of the chaos synchronization approaches, the master-slave or drive-response formalism is used. If a particular chaotic system is called the master or drive system and another chaotic system is called the slave or response system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

In the last two decades, various schemes have been successfully applied for chaos synchronization such as PC method [2], OGY method [9], active control method [10-14], adaptive control method [15-18], time-delay feedback method [19], backstepping design method [20-22], sampled-data feedback method [23], sliding mode control method [24-27], etc.
In this paper, we apply adaptive control method to derive new results for the global chaos synchronization of identical hyperchaotic Lorenz systems ([28], 2007), identical hyperchaotic Lü systems ([29], 2006) and non-identical hyperchaotic Lorenz and hyperchaotic Lü systems. We assume that the parameters of the master and slave systems are unknown and we devise adaptive synchronizing schemes using the estimates of the parameters for both master and slave systems.

This paper has been organized as follows. In Section 2, we give a description of hyperchaotic Lorenz and hyperchaotic Lü systems. In Section 3, we discuss the adaptive synchronization of identical hyperchaotic Lorenz systems. In Section 4, we discuss the adaptive synchronization of identical hyperchaotic Lü systems. In Section 5, we discuss the adaptive synchronization of non-identical hyperchaotic Lorenz and hyperchaotic Lü systems. In Section 6, we summarize the main results obtained in this paper.

2. SYSTEMS DESCRIPTION

The hyperchaotic Lorenz system ([28], 2007) is described by

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1 x_3 + rx_1 - x_2 \\
\dot{x}_3 &= x_1 x_2 - bx_3 \\
\dot{x}_4 &= -x_1 x_3 + dx_4
\end{align*}
\]  

(1)

where \(x_1, x_2, x_3, x_4\) are the state variables and \(a, b, r, d\) are positive constant parameters of the system.

The system (1) is hyperchaotic when the parameter values are taken as

\[a = 10, \quad r = 28, \quad b = 8/3 \quad \text{and} \quad d = 1.3.\]

The state orbits of the hyperchaotic Lorenz system (1) are shown in Figure 1.

The hyperchaotic Lü system ([29], 2006) is described by

\[
\begin{align*}
\dot{x}_1 &= \alpha(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1 x_3 + \gamma x_2 \\
\dot{x}_3 &= x_1 x_2 - \beta x_3 \\
\dot{x}_4 &= x_1 x_3 + \delta x_4
\end{align*}
\]  

(2)

where \(x_1, x_2, x_3, x_4\) are the state variables and \(\alpha, \beta, \gamma, \delta\) are positive constant parameters of the system.

The system (2) is hyperchaotic when the parameter values are taken as

\[\alpha = 36, \quad \beta = 3, \quad \gamma = 20 \quad \text{and} \quad \delta = 1.3.\]

The state orbits of the hyperchaotic Lü system (2) are shown in Figure 2.
Figure 1. State Orbits of the Hyperchaotic Lorenz System

Figure 2. State Orbits of the Hyperchaotic Lü System
3. ADAPTIVE SYNCHRONIZATION OF IDENTICAL HYPERCHAOTIC LORENZ SYSTEMS

3.1 Theoretical Results

In this section, we discuss the adaptive synchronization of identical hyperchaotic Lorenz systems ([28], 2007), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Lorenz dynamics described by

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1x_3 + rx_1 - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{x}_4 &= -x_1x_3 + dx_4
\end{align*}
\] (3)

where \(x_1, x_2, x_3, x_4\) are the states and \(a, b, r, d\) are unknown real constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Lorenz dynamics described by

\[
\begin{align*}
\dot{y}_1 &= a(y_2 - y_1) + y_4 + u_1 \\
\dot{y}_2 &= -y_1y_3 + ry_1 - y_2 + u_2 \\
\dot{y}_3 &= y_1y_2 - by_3 + u_3 \\
\dot{y}_4 &= -y_1y_3 + dy_3 + u_4
\end{align*}
\] (4)

where \(y_1, y_2, y_3, y_4\) are the states and \(u_1, u_2, u_3, u_4\) are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

\[
e_i = y_i - x_i, \quad (i = 1, 2, 3, 4)
\] (5)

The error dynamics is easily obtained as

\[
\begin{align*}
\dot{e}_1 &= a(e_2 - e_1) + e_4 + u_1 \\
\dot{e}_2 &= -e_1e_3 - y_1y_3 - x_1x_3 + u_2 \\
\dot{e}_3 &= -be_3 + y_1y_2 - x_1x_2 + u_3 \\
\dot{e}_4 &= de_4 - y_1y_3 + x_1x_3 + u_4
\end{align*}
\] (6)

Let us now define the adaptive control functions

\[
\begin{align*}
u_1(t) &= -\hat{a}(e_2 - e_1) - e_4 - k_1e_1 \\
u_2(t) &= -\hat{r}e_1 + e_2 + y_1y_3 - x_1x_3 - k_2e_2 \\
u_3(t) &= \hat{b}e_3 - y_1y_2 + x_1x_2 - k_3e_3 \\
u_4(t) &= -\hat{d}e_4 + y_1y_3 - x_1x_3 - k_4e_4
\end{align*}
\] (7)
where \( \hat{a}, \hat{b}, \hat{r} \) and \( \hat{d} \) are estimates of \( a, b, r \) and \( d \), respectively, and \( k_i, (i = 1, 2, 3, 4) \) are positive constants.

Substituting (7) into (6), the error dynamics simplifies to

\[
\begin{align*}
\dot{e}_1 &= (a - \hat{a})(e_2 - e_1) - k_1 e_1 \\
\dot{e}_2 &= (r - \hat{r})e_1 - k_2 e_2 \\
\dot{e}_3 &= -(b - \hat{b})e_3 - k_3 e_3 \\
\dot{e}_4 &= (d - \hat{d})e_4 - k_4 e_4 
\end{align*}
\]

(8)

Let us now define the parameter estimation errors as

\[
\begin{align*}
e_a &= a - \hat{a}, & e_b &= b - \hat{b}, & e_r &= r - \hat{r} & \text{and} & e_d &= d - \hat{d}.
\end{align*}
\]

(9)

Substituting (9) into (8), we obtain the error dynamics as

\[
\begin{align*}
\dot{e}_1 &= e_a(e_2 - e_1) - k_1 e_1 \\
\dot{e}_2 &= e_e e_1 - k_2 e_2 \\
\dot{e}_3 &= -e_e e_3 - k_3 e_3 \\
\dot{e}_4 &= e_d e_4 - k_4 e_4
\end{align*}
\]

(10)

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

\[
V(e_1, e_2, e_3, e_4, e_a, e_b, e_r, e_d) = \frac{1}{2} \left( e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_r^2 + e_d^2 \right)
\]

(11)

which is a positive definite function on \( \mathbb{R}^8 \).

We also note that

\[
\begin{align*}
\dot{e}_a &= -\dot{\hat{a}}, & \dot{e}_b &= -\dot{\hat{b}}, & \dot{e}_r &= -\dot{\hat{r}} & \text{and} & \dot{e}_d &= -\dot{\hat{d}}
\end{align*}
\]

(12)

Differentiating (11) along the trajectories of (10) and using (12), we obtain

\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a \left[ e_1 (e_2 - e_1) - \dot{\hat{a}} \right] \\
+ e_b \left[ -e_3 - \dot{\hat{b}} \right] + e_r \left[ e_4 e_2 - \dot{\hat{r}} \right] + e_d \left[ e_d^2 - \dot{\hat{d}} \right]
\]

(13)
In view of Eq. (13), the estimated parameters are updated by the following law:

\[
\begin{align*}
\dot{\hat{a}} &= e_i (e_z - e_i) + k_5 e_a \\
\dot{\hat{b}} &= -e_i^2 + k_6 e_b \\
\dot{\hat{r}} &= e_i e_z + k_7 e_r \\
\dot{\hat{d}} &= e_i^2 + k_8 e_d
\end{align*}
\]  

(14)

where \( k_5, k_6, k_7 \) and \( k_8 \) are positive constants.

Substituting (14) into (12), we obtain

\[
\dot{V} = -k_5 e_i^2 - k_6 e_b^2 - k_7 e_r^2 - k_8 e_d^2 - k_6 e_b^2 - k_7 e_r^2 - k_8 e_d^2
\]

(15)

which is a negative definite function on \( \mathbb{R}^8 \).

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error \( e_i, (i = 1, 2, 3, 4) \) and the parameter estimation error \( e_a, e_b, e_r, e_d \) decay to zero exponentially with time.

Hence, we have proved the following result.

**Theorem 1.** The identical hyperchaotic Lorenz systems (3) and (4) with unknown parameters are globally and exponentially synchronized by the adaptive control law (7), where the update law for the parameter estimates is given by (14) and \( k_i, (i = 1, 2, \ldots, 8) \) are positive constants.

3.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step \( h = 10^{-6} \) is used to solve the hyperchaotic systems (3) and (4) with the adaptive control law (14) and the parameter update law (14) using MATLAB. We take \( k_i = 2 \) for \( i = 1, 2, \ldots, 8 \).

For the hyperchaotic Lorenz systems (3) and (4), the parameter values are taken as

\[
a = 10, \quad r = 28, \quad b = 8/3 \quad \text{and} \quad d = 1.3.
\]

Suppose that the initial values of the parameter estimates are

\[
\hat{a}(0) = 20, \quad \hat{b}(0) = 2, \quad \hat{r}(0) = 12, \quad \hat{d}(0) = 15.
\]

The initial values of the master system (3) are taken as

\[
x_1(0) = 8, \quad x_2(0) = 14, \quad x_3(0) = 36, \quad x_4(0) = 25.
\]

The initial values of the slave system (4) are taken as

\[
y_1(0) = 28, \quad y_2(0) = 6, \quad y_3(0) = 20, \quad y_4(0) = 12.
\]
Figure 3 depicts the complete synchronization of the identical hyperchaotic Lorenz systems (3) and (4). Figure 4 shows that the estimated values of the parameters, viz. $\hat{a}, \hat{b}, \hat{r}$ and $\hat{d}$ converge to the system parameters $a = 10$, $r = 28$, $b = 8/3$ and $d = 1.3$. 

Figure 3. Complete Synchronization of the Hyperchaotic Lorenz Systems

Figure 4. Parameter Estimates $\hat{a}, \hat{b}, \hat{r}, \hat{d}$
4. ADAPTIVE SYNCHRONIZATION OF IDENTICAL HYPERCHAOTIC LÜ SYSTEMS

4.1 Theoretical Results

In this section, we discuss the adaptive synchronization of identical hyperchaotic Lü systems ([29], 2006), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Lü dynamics described by

\[
\begin{align*}
\dot{x}_1 &= \alpha(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1x_3 + \gamma x_2 \\
\dot{x}_3 &= x_1x_2 - \beta x_3 \\
\dot{x}_4 &= x_1x_3 + \delta x_4
\end{align*}
\]

(16)

where \(x_1, x_2, x_3, x_4\) are the states and \(\alpha, \beta, \gamma, \delta\) are unknown real constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Lü dynamics described by

\[
\begin{align*}
\dot{y}_1 &= \alpha(y_2 - y_1) + y_4 + u_1 \\
\dot{y}_2 &= -y_1y_3 + \gamma y_2 + u_2 \\
\dot{y}_3 &= y_1y_2 - \beta y_3 + u_3 \\
\dot{y}_4 &= y_1y_3 + \delta y_4 + u_4
\end{align*}
\]

(17)

where \(y_1, y_2, y_3, y_4\) are the states and \(u_1, u_2, u_3, u_4\) are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

\[e_i = y_i - x_i, \quad (i = 1, 2, 3, 4)\]

(18)

The error dynamics is easily obtained as

\[
\begin{align*}
\dot{e}_1 &= \alpha(e_2 - e_1) + e_4 + u_1 \\
\dot{e}_2 &= \gamma e_2 - y_1y_3 + x_1x_3 + u_2 \\
\dot{e}_3 &= -\beta e_3 + y_1y_2 - x_1x_2 + u_3 \\
\dot{e}_4 &= \delta e_4 + y_1y_3 - x_1x_3 + u_4
\end{align*}
\]

(19)

Let us now define the adaptive control functions

\[
\begin{align*}
u_1(t) &= -\hat{\alpha}(e_2 - e_1) - e_4 - k_1e_1 \\
u_2(t) &= -\hat{\gamma}e_2 + y_1y_3 - x_1x_3 - k_2e_2 \\
u_3(t) &= \hat{\beta}e_3 - y_1y_2 + x_1x_2 - k_3e_3 \\
u_4(t) &= -\hat{\delta}e_4 - y_1y_3 + x_1x_3 - k_4e_4
\end{align*}
\]

(20)
where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\delta}$ are estimates of $\alpha, \beta, \gamma$ and $\delta$, respectively, and $k_i, (i = 1, 2, 3, 4)$ are positive constants.

Substituting (20) into (19), the error dynamics simplifies to

$$
\begin{align*}
\dot{e}_1 &= (\alpha - \hat{\alpha})(e_2 - e_1) - k_1 e_1 \\
\dot{e}_2 &= (\gamma - \hat{\gamma})e_2 - k_2 e_2 \\
\dot{e}_3 &= -(\beta - \hat{\beta})e_3 - k_3 e_3 \\
\dot{e}_4 &= (\delta - \hat{\delta})e_4 - k_4 e_4
\end{align*}
$$

(21)

Let us now define the parameter estimation errors as

$$
\begin{align*}
e_a &= \alpha - \hat{\alpha}, \quad e_\beta = \beta - \hat{\beta}, \quad e_\gamma = \gamma - \hat{\gamma} \quad \text{and} \quad e_\delta = \delta - \hat{\delta}.
\end{align*}
$$

(22)

Substituting (22) into (21), we obtain the error dynamics as

$$
\begin{align*}
\dot{e}_1 &= e_a (e_2 - e_1) - k_1 e_1 \\
\dot{e}_2 &= e_\beta e_2 - k_2 e_2 \\
\dot{e}_3 &= -e_\beta e_3 - k_3 e_3 \\
\dot{e}_4 &= e_\delta e_4 - k_4 e_4
\end{align*}
$$

(23)

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$
V(e_1, e_2, e_3, e_4, e_a, e_\beta, e_\gamma, e_\delta) = \frac{1}{2} \left( e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2 \right)
$$

(24)

which is a positive definite function on $\mathbb{R}^8$.

We also note that

$$
\begin{align*}
\dot{e}_a &= -\dot{\hat{\alpha}}, \quad \dot{e}_\beta = -\dot{\hat{\beta}}, \quad \dot{e}_\gamma = -\dot{\hat{\gamma}} \quad \text{and} \quad \dot{e}_\delta = -\dot{\hat{\delta}}
\end{align*}
$$

(25)

Differentiating (24) along the trajectories of (23) and using (25), we obtain

$$
\begin{align*}
\dot{V} &= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a \left[ e_1 (e_2 - e_1) - \dot{\hat{\alpha}} \right] \\
&\quad + e_\beta \left[ -e_2^2 - \dot{\hat{\beta}} \right] + e_\gamma \left[ e_2^2 - \dot{\hat{\gamma}} \right] + e_\delta \left[ e_4^2 - \dot{\hat{\delta}} \right]
\end{align*}
$$

(26)
In view of Eq. (26), the estimated parameters are updated by the following law:

\[
\begin{align*}
\dot{\hat{\alpha}} &= e_1(e_2 - e_1) + k_5 e_\alpha \\
\dot{\hat{\beta}} &= -e_3 + k_6 e_\beta \\
\dot{\hat{\gamma}} &= e_2 + k_7 e_\gamma \\
\dot{\hat{\delta}} &= e_4 + k_8 e_\delta
\end{align*}
\]

where \(k_5, k_6, k_7,\) and \(k_8\) are positive constants.

Substituting (27) into (26), we obtain

\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_\alpha^2 - k_6 e_\beta^2 - k_7 e_\gamma^2 - k_8 e_\delta^2
\]

which is a negative definite function on \(\mathbb{R}^8\).

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error \(e_i, (i = 1, 2, 3, 4)\) and the parameter estimation error \(e_\alpha, e_\beta, e_\gamma, e_\delta\) decay to zero exponentially with time.

Hence, we have proved the following result.

**Theorem 2.** The identical hyperchaotic Lü systems (16) and (17) with unknown parameters are globally and exponentially synchronized by the adaptive control law (20), where the update law for the parameter estimates is given by (27) and \(k_i, (i = 1, 2, \ldots, 8)\) are positive constants.

**4.2 Numerical Results**

For the numerical simulations, the fourth-order Runge-Kutta method with time-step \(h = 10^{-6}\) is used to solve the hyperchaotic systems (16) and (17) with the adaptive control law (14) and the parameter update law (27) using MATLAB. We take \(k_i = 2\) for \(i = 1, 2, \ldots, 8\).

For the hyperchaotic Lü systems (16) and (17), the parameter values are taken as

\[
\alpha = 36, \quad \beta = 3, \quad \gamma = 20 \quad \text{and} \quad \delta = 1.3.
\]

Suppose that the initial values of the parameter estimates are

\[
\hat{\alpha}(0) = 2, \quad \hat{\beta}(0) = 8, \quad \hat{\gamma}(0) = 4, \quad \hat{\delta}(0) = 5
\]

The initial values of the master system (16) are taken as

\[
x_1(0) = 18, \quad x_2(0) = 24, \quad x_3(0) = 14, \quad x_4(0) = 30.
\]

The initial values of the slave system (17) are taken as

\[
y_1(0) = 20, \quad y_2(0) = 7, \quad y_3(0) = 32, \quad y_4(0) = 9.
\]
Figure 5 depicts the complete synchronization of the identical hyperchaotic Lü systems (16) and (17). Figure 6 shows that the estimated values of the parameters, viz. $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\delta}$ converge to the system parameters $\alpha = 36, \beta = 3, \gamma = 20$ and $\delta = 1.3$. 

![Figure 5. Complete Synchronization of the Hyperchaotic Lü Systems](image)

![Figure 6. Parameter Estimates $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$](image)
5. ADAPTIVE SYNCHRONIZATION OF NON-IDENTICAL HYPERCHAOTIC LORENZ AND HYPERCHAOTIC LÜ SYSTEMS

5.1 Theoretical Results

In this section, we discuss the adaptive synchronization of non-identical hyperchaotic Lorenz system ([28], 2007) and hyperchaotic Lü system ([29], 2006), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Lorenz dynamics described by

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1x_3 + rx_1 - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{x}_4 &= -x_1x_3 + dx_4
\end{align*}
\]

(29)

where \( x_1, x_2, x_3, x_4 \) are the states and \( a, b, c, d \) are unknown real constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Lü dynamics described by

\[
\begin{align*}
\dot{y}_1 &= \alpha(y_2 - y_1) + y_4 + u_1 \\
\dot{y}_2 &= -y_1y_3 + \gamma y_2 + u_2 \\
\dot{y}_3 &= y_1y_2 - \beta y_3 + u_3 \\
\dot{y}_4 &= y_1y_3 + \delta y_4 + u_4
\end{align*}
\]

(30)

where \( y_1, y_2, y_3, y_4 \) are the states, \( \alpha, \beta, \gamma, \delta \) are unknown real constant parameters of the system and \( u_1, u_2, u_3, u_4 \) are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

\[
e_i = y_i - x_i, \quad (i = 1, 2, 3, 4)
\]

(31)

The error dynamics is easily obtained as

\[
\begin{align*}
\dot{e}_1 &= \alpha(y_2 - y_1) - a(x_2 - x_1) + e_4 + u_1 \\
\dot{e}_2 &= -y_1y_3 + \gamma y_2 - y_1y_3 + x_1x_3 + u_2 \\
\dot{e}_3 &= -\beta y_3 + bx_3 + y_1y_2 - x_1x_2 + u_3 \\
\dot{e}_4 &= \delta y_4 - dx_3 + y_1y_3 + x_1x_3 + u_4
\end{align*}
\]

(32)
Let us now define the adaptive control functions

\[
\begin{align*}
    u_1(t) &= -\alpha(y_2 - y_1) + \hat{a}(x_2 - x_1) + e_1, \\
    u_2(t) &= -\gamma y_2 + \hat{r}x_2 - x_1 + y_1 y_3 - x_1 x_3 - k_2 e_2, \\
    u_3(t) &= -\beta y_3 + \hat{b}x_3 - y_1 y_2 + x_1 x_2 - k_3 e_3, \\
    u_4(t) &= -\delta y_4 - \hat{d}x_4 - y_1 y_3 - x_1 x_3 - k_4 e_4
\end{align*}
\]

where \( \hat{a}, \hat{b}, \hat{r}, \hat{d}, \hat{\alpha}, \hat{\beta}, \hat{\gamma} \) and \( \hat{\delta} \) are estimates of \( a, b, r, d, \alpha, \beta, \gamma \) and \( \delta \), respectively, and \( k_i, (i = 1, 2, 3, 4) \) are positive constants.

Substituting (33) into (32), the error dynamics simplifies to

\[
\begin{align*}
    \dot{e}_1 &= (\alpha - \hat{\alpha})(y_2 - y_1) - (a - \hat{a})(x_2 - x_1) - k_1 e_1, \\
    \dot{e}_2 &= (\gamma - \hat{\gamma})y_2 - (r - \hat{r})x_1 - k_2 e_2, \\
    \dot{e}_3 &= -(\beta - \hat{\beta})y_3 + (b - \hat{b})x_3 - k_3 e_3, \\
    \dot{e}_4 &= -(\delta - \hat{\delta})y_4 - (d - \hat{d})x_4 - k_4 e_4
\end{align*}
\]

Let us now define the parameter estimation errors as

\[
\begin{align*}
    e_a &= a - \hat{a}, & e_b &= b - \hat{b}, & e_r &= r - \hat{r}, & e_d &= d - \hat{d}, \\
    e_\alpha &= \alpha - \hat{\alpha}, & e_\beta &= \beta - \hat{\beta}, & e_\gamma &= \gamma - \hat{\gamma}, & e_\delta &= \delta - \hat{\delta}
\end{align*}
\]

Substituting (35) into (32), we obtain the error dynamics as

\[
\begin{align*}
    \dot{e}_1 &= e_a(y_2 - y_1) - e_a x_2 - x_1 - k_1 e_1, \\
    \dot{e}_2 &= e_r y_2 - e_a x_1 - k_2 e_2, \\
    \dot{e}_3 &= -e_\beta y_3 + e_a x_3 - k_3 e_3, \\
    \dot{e}_4 &= e_\delta y_4 - e_d x_4 - k_4 e_4
\end{align*}
\]

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

\[
V = \frac{1}{2} \left( e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2 \right)
\]

which is a positive definite function on \( \mathbb{R}^{12} \).

We also note that

\[
\begin{align*}
    \dot{e}_a &= -\dot{\hat{a}}, & \dot{e}_b &= -\dot{\hat{b}}, & \dot{e}_r &= -\dot{\hat{r}}, & \dot{e}_d &= -\dot{\hat{d}}, & \dot{e}_\alpha &= -\dot{\hat{\alpha}}, & \dot{e}_\beta &= -\dot{\hat{\beta}}, & \dot{e}_\gamma &= -\dot{\hat{\gamma}}, & \dot{e}_\delta &= -\dot{\hat{\delta}}
\end{align*}
\]
Differentiating (37) along the trajectories of (36) and using (38), we obtain

\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a \left[ -e_1 (x_2 - x_1) - \dot{\alpha} \right] + e_b \left[ e_3 x_3 - \dot{\beta} \right] \\
+ e_r \left[ -e_2 x_1 - \dot{\gamma} \right] + e_d \left[ -e_4 x_4 - \dot{\delta} \right] + e_e \left[ e_1 (y_2 - y_1) - \dot{\alpha} \right] \\
+ e_\theta \left[ -e_3 y_3 - \dot{\beta} \right] + e_\phi \left[ e_2 y_2 - \dot{\gamma} \right] + \alpha \left[ e_4 y_4 - \dot{\delta} \right]
\]

(39)

In view of Eq. (39), the estimated parameters are updated by the following law:

\[
\dot{\hat{a}} = -e_1 (x_2 - x_1) + k_5 e_a, \quad \dot{\hat{b}} = e_3 x_3 + k_6 e_b, \quad \dot{\hat{c}} = e_4 x_4 + k_7 e_c, \\
\dot{\hat{\alpha}} = e_1 (y_2 - y_1) + k_9 e_a, \quad \dot{\hat{\beta}} = e_3 y_3 + k_{10} e_b, \\
\dot{\hat{\gamma}} = e_2 y_2 + k_1 e_\gamma, \quad \dot{\hat{\delta}} = e_4 y_4 + k_2 e_\delta
\]

(40)

where \(k_5, k_6, k_7\) and \(k_8\) are positive constants.

Substituting (40) into (39), we obtain

\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + k_9 e_a^2 - k_{10} e_b^2 - k_1 e_\gamma^2 - k_2 e_\delta^2 - k_3 e_\phi^2 - k_4 e_\theta^2 - k_5 e_a - k_6 e_b - k_7 e_c - k_9 e_d - k_{10} e_\alpha - k_1 e_\beta - k_2 e_\gamma - k_3 e_\delta
\]

(41)

which is a negative definite function on \(\mathbb{R}^{12}\).

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error \(e_i, (i = 1, 2, 3, 4)\) and the parameter estimation error decay to zero exponentially with time.

Hence, we have proved the following result.

**Theorem 3.** The non-identical hyperchaotic Lorenz system (29) and hyperchaotic Lü system (30) with unknown parameters are globally and exponentially synchronized by the adaptive control law (33), where the update law for the parameter estimates is given by (40) and \(k_i, (i = 1, 2, \ldots, 12)\) are positive constants.

**5.2 Numerical Results**

For the numerical simulations, the fourth-order Runge-Kutta method with time-step \(h = 10^{-6}\) is used to solve the hyperchaotic systems (29) and (30) with the adaptive control law (27) and the parameter update law (40) using MATLAB. We take \(k_i = 2\) for \(i = 1, 2, \ldots, 12\).

For the hyperchaotic Lorenz system (29) and hyperchaotic Lü system (30), the parameter values are taken as

\[
a = 10, \quad b = 8/3, \quad r = 28, \quad d = 1.3, \quad \alpha = 36, \quad \beta = 3, \quad \gamma = 20 \quad \text{and} \quad \delta = 1.3.
\]

(42)
Suppose that the initial values of the parameter estimates are
\[
\hat{a}(0) = 7, \quad \hat{b}(0) = 10, \quad \hat{r}(0) = 1, \quad \hat{d}(0) = 4, \quad \hat{\alpha}(0) = 2, \quad \hat{\beta}(0) = 8, \quad \hat{\gamma}(0) = 4, \quad \hat{\delta}(0) = 5.
\]
The initial values of the master system (29) are taken as
\[
x_1(0) = 12, \quad x_2(0) = 10, \quad x_3(0) = 32, \quad x_4(0) = 27.
\]
The initial values of the slave system (30) are taken as
\[
y_1(0) = 4, \quad y_2(0) = 27, \quad y_3(0) = 12, \quad y_4(0) = 16.
\]
Figure 7 depicts the complete synchronization of the non-identical hyperchaotic Lorenz and hyperchaotic Lü systems. Figure 8 shows that the estimated values of the parameters, viz. \(\hat{a}, \hat{b}, \hat{r}, \hat{d}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}\) and \(\hat{\delta}\) converge to the original values of the parameters given in (42).
5. CONCLUSIONS

In this paper, we have applied adaptive control method for the global chaos synchronization of identical hyperchaotic Lorenz systems (2007), identical hyperchaotic Lü systems (2006) and non-identical hyperchaotic Lorenz and hyperchaotic Lü systems with unknown parameters. The adaptive synchronization results derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is a very effective and convenient for achieving chaos synchronization for the uncertain hyperchaotic systems discussed in this paper. Numerical simulations are shown to demonstrate the effectiveness of the adaptive synchronization schemes derived in this paper for the synchronization of identical and non-identical uncertain hyperchaotic Lorenz and hyperchaotic Lü systems.

REFERENCES


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