# GLOBAL CHAOS SYNCHRONIZATION OF UNCERTAIN LORENZ-STENFLO AND QI 4-D CHAOTIC SYSTEMS BY ADAPTIVE CONTROL

Sundarapandian Vaidyanathan[1]

[1]Research and Development Centre, Vel Tech Dr. RR & Dr. SR Technical University
Avadi, Chennai-600 062, Tamil Nadu, INDIA
sundarvtu@gmail.com

***ABSTRACT***

*In this paper, we apply adaptive control method to derive new results for the global chaos synchronization of 4-D chaotic systems, viz. identical Lorenz-Stenflo(LS) systems (Stenflo, 2001), identical Qi systems (Qi, Chen and Du, 2005) and non-identical LS and Qi systems. In this paper, we shall assume that the parameters of both master and slave systems are unknown and we devise adaptive control schemes for synchronization using the estimates of parameters for both master and slave systems. Our adaptive synchronization schemes derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is very effective and convenient to synchronize identical and non-identical LS and Qi systems. Numerical simulations are shown to demonstrate the effectiveness of the proposed adaptive synchronization schemes for the identical and non-identical, uncertain LS and Qi 4-D chaotic systems.*

***KEYWORDS***

*Adaptive Control, Chaos, Synchronization, Lorenz-Stenflo System, Qi System.*

## 1. INTRODUCTION

Chaotic systems are nonlinear dynamical systems that are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the *butterfly effect* [1].

Chaos theory has been applied to a variety of fields such as physical systems [2-4], ecological systems [5-6], chemical reactor [7], secure communications [8-10], etc. Since the seminal work by Pecora and Carroll ([11], 1990), chaos synchronization problem has been studied extensively and intensively in the chaos literature [11-29].

In most of the chaos synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

The problem of chaos synchronization is related to the observer problem in control theory. In general, the designed controller makes the trajectories of the slave system to track the trajectories of the master system.

Progress in the research activities on chaos synchronization has given birth to various methods of synchronization such as PC method [11], OGY method [12], active control method [13-17], adaptive control method [18-20], time-delay feedback method [21], backstepping design method [22-23], sampled-data feedback method [24], sliding mode control method [25-27], etc.

In this paper, we apply adaptive control method to derive new results for the global chaos synchronization of identical Lorenz-Stenflo systems ([28], 2001), identical Qi systems ([29], 2005) and non-identical LS and Qi systems. We assume that the parameters of the master and slave systems are unknown and we devise adaptive synchronizing schemes using the estimates of the parameters for both master and slave systems.

This paper has been organized as follows. In Section 2, we give a description of LS and Qi 4-D chaotic systems. In Section 3, we discuss the adaptive synchronization of identical LS systems. In Section 4, we discuss the adaptive synchronization of identical Qi systems. In Section 5, we discuss the adaptive synchronization of LS and Qi chaotic systems. In Section 6, we summarize the main results obtained in this paper.

## 2. SYSTEMS DESCRIPTION

The Lorenz-Stenflo system ([28], 2001) is described by

$$\begin{aligned}
\dot{x}_1 &= \alpha(x_2 - x_1) + \gamma x_4 \\
\dot{x}_2 &= x_1(r - x_3) - x_2 \\
\dot{x}_3 &= x_1 x_2 - \beta x_3 \\
\dot{x}_4 &= -x_1 - \alpha x_4
\end{aligned} \tag{1}$$

where $x_1, x_2, x_3, x_4$ are the state variables and $\alpha, \beta, \gamma, r$ are positive constant parameters of the system.

The LS system (1) is chaotic when the parameter values are taken as

$$\alpha = 2.0, \quad \beta = 0.7, \quad \gamma = 1.5 \quad \text{and} \quad r = 26.0.$$

The state orbits of the chaotic LS system (1) are shown in Figure 1.

The Qi system ([29], 2005) is described by

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_2 x_3 x_4 \\
\dot{x}_2 &= b(x_1 + x_2) - x_1 x_3 x_4 \\
\dot{x}_3 &= -c x_3 + x_1 x_2 x_4 \\
\dot{x}_4 &= -d x_4 + x_1 x_2 x_3
\end{aligned} \tag{2}$$

where $x_1, x_2, x_3, x_4$ are the state variables and $a, b, c, d$ are positive constant parameters of the system.

The system (2) is hyperchaotic when the parameter values are taken as

$$a = 30, \quad b = 10, \quad c = 1 \quad \text{and} \quad d = 10.$$

The state orbits of the chaotic Qi system (2) are shown in Figure 2.

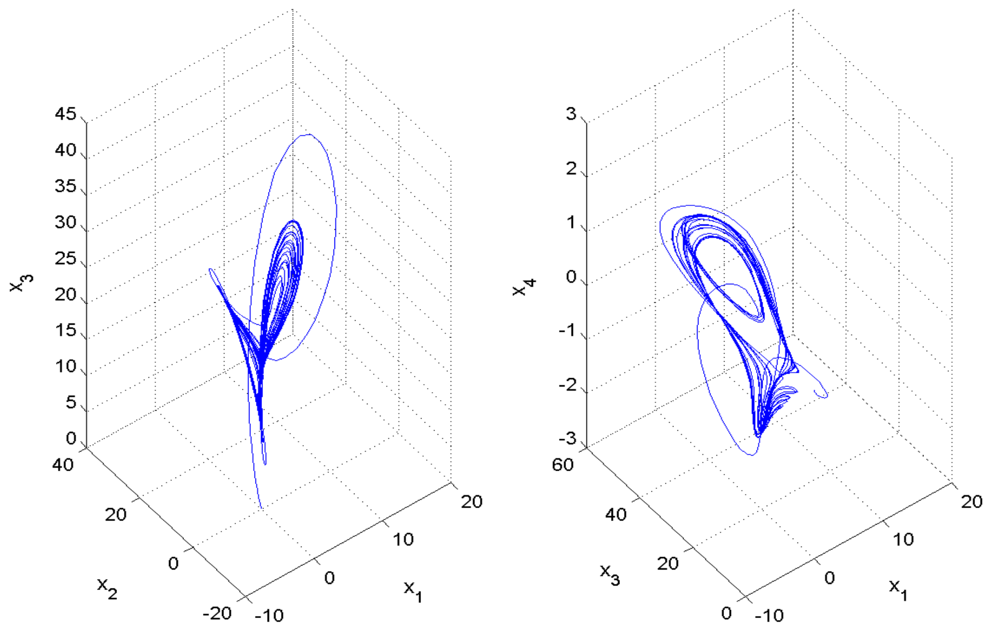


Figure 1. State Orbits of the Lorenz-Stenflo Chaotic System

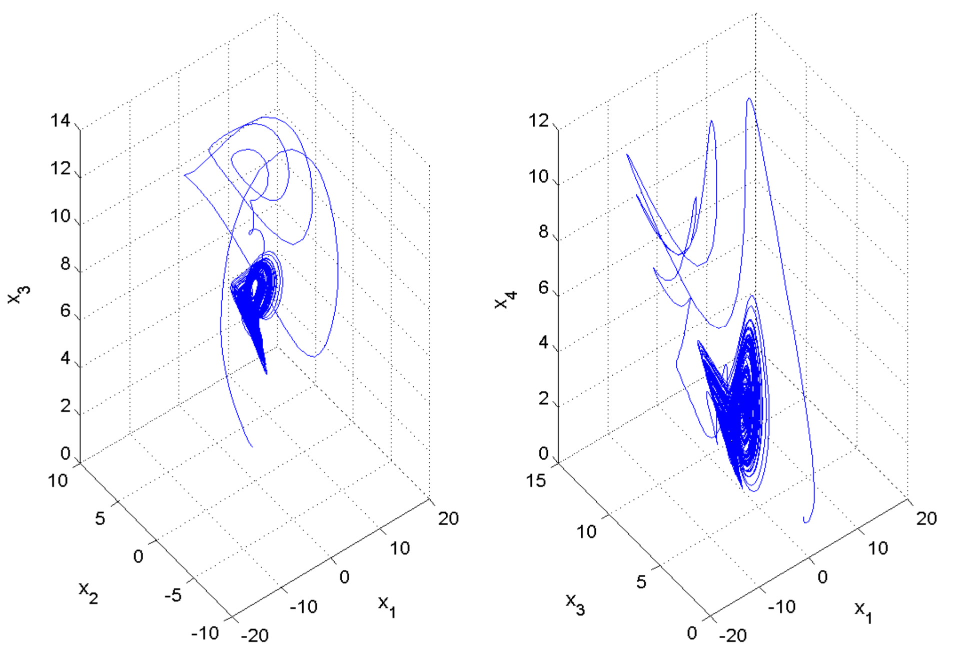


Figure 2. State Orbits of the Qi Chaotic System

# 3. ADAPTIVE SYNCHRONIZATION OF IDENTICAL LORENZ-STENFLO SYSTEMS

## 3.1 Theoretical Results

In this section, we discuss the adaptive synchronization of identical Lorenz-Stenflo systems ([28], 2001), where the parameters of the master and slave systems are unknown.

As the master system, we consider the LS dynamics described by

$$\begin{aligned}
\dot{x}_1 &= \alpha(x_2 - x_1) + \gamma x_4 \\
\dot{x}_2 &= x_1(r - x_3) - x_2 \\
\dot{x}_3 &= x_1 x_2 - \beta x_3 \\
\dot{x}_4 &= -x_1 - \alpha x_4
\end{aligned} \tag{3}$$

where $x_1, x_2, x_3, x_4$ are the states and $\alpha, \beta, \gamma, r$ are unknown real constant parameters of the system.

As the slave system, we consider the controlled LS dynamics described by

$$\begin{aligned}
\dot{y}_1 &= \alpha(y_2 - y_1) + \gamma y_4 + u_1 \\
\dot{y}_2 &= y_1(r - y_3) - y_2 + u_2 \\
\dot{y}_3 &= y_1 y_2 - \beta y_3 + u_3 \\
\dot{y}_4 &= -y_1 - \alpha y_4 + u_4
\end{aligned} \tag{4}$$

where $y_1, y_2, y_3, y_4$ are the states and $u_1, u_2, u_3, u_4$ are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

$$e_i = y_i - x_i, \quad (i = 1, 2, 3, 4) \tag{5}$$

The error dynamics is easily obtained as

$$\begin{aligned}
\dot{e}_1 &= \alpha(e_2 - e_1) + \gamma e_4 + u_1 \\
\dot{e}_2 &= -e_2 + r(e_1 - y_1 y_3 + x_1 x_3) + u_2 \\
\dot{e}_3 &= -\beta e_3 + y_1 y_2 - x_1 x_2 + u_3 \\
\dot{e}_4 &= -e_1 - \alpha e_4 + u_4
\end{aligned} \tag{6}$$

Let us now define the adaptive control functions

$$\begin{aligned}
u_1(t) &= -\hat{\alpha}(e_2 - e_1) - \hat{\gamma} e_4 - k_1 e_1 \\
u_2(t) &= e_2 - \hat{r}(e_1 - y_1 y_3 + x_1 x_3) - k_2 e_2 \\
u_3(t) &= \hat{\beta} e_3 - y_1 y_2 + x_1 x_2 - k_3 e_3 \\
u_4(t) &= e_1 + \hat{\alpha} e_4 - k_4 e_4
\end{aligned} \tag{7}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{r}$ are estimates of $\alpha, \beta, \gamma$ and $r$, respectively, and $k_i, (i = 1, 2, 3, 4)$ are positive constants.

Substituting (7) into (6), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (\alpha - \hat{\alpha})(e_2 - e_1) + (\gamma - \hat{\gamma})e_4 - k_1 e_1 \\ \dot{e}_2 &= (r - \hat{r})(e_1 - y_1 y_3 + x_1 x_3) - k_2 e_2 \\ \dot{e}_3 &= -(\beta - \hat{\beta})e_3 - k_3 e_3 \\ \dot{e}_4 &= -(\alpha - \hat{\alpha})e_4 - k_4 e_4 \end{aligned} \tag{8}$$

Let us now define the parameter estimation errors as

$$e_\alpha = \alpha - \hat{\alpha}, \quad e_\beta = \beta - \hat{\beta}, \quad e_\gamma = \gamma - \hat{\gamma} \quad \text{and} \quad e_r = r - \hat{r}. \tag{9}$$

Substituting (9) into (8), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_\alpha (e_2 - e_1) + e_\gamma e_4 - k_1 e_1 \\ \dot{e}_2 &= e_r (e_1 - y_1 y_3 + x_1 x_3) - k_2 e_2 \\ \dot{e}_3 &= -e_\beta e_3 - k_3 e_3 \\ \dot{e}_4 &= -e_\alpha e_4 - k_4 e_4 \end{aligned} \tag{10}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V(e_1, e_2, e_3, e_4, e_\alpha, e_\beta, e_\gamma, e_r) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_r^2\right) \tag{11}$$

which is a positive definite function on $R^8$.

We also note that

$$\dot{e}_\alpha = -\dot{\hat{\alpha}}, \; \dot{e}_\beta = -\dot{\hat{\beta}}, \; \dot{e}_\gamma = -\dot{\hat{\gamma}} \quad \text{and} \quad \dot{e}_r = -\dot{\hat{r}} \tag{12}$$

Differentiating (11) along the trajectories of (10) and using (12), we obtain

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_\alpha \left[ e_1 (e_2 - e_1) - e_4^2 - \dot{\hat{\alpha}} \right] + e_\beta \left[ -e_3^2 - \dot{\hat{\beta}} \right] \\ &+ e_\gamma \left[ e_1 e_4 - \dot{\hat{\gamma}} \right] + e_r \left[ e_2 (e_1 - y_1 y_3 + x_1 x_3) - \dot{\hat{r}} \right] \end{aligned} \tag{13}$$

In view of Eq. (13), the estimated parameters are updated by the following law:

$$\begin{aligned}\dot{\hat{\alpha}} &= e_1(e_2 - e_1) - e_4^2 + k_5 e_\alpha \\ \dot{\hat{\beta}} &= -e_3^2 + k_6 e_\beta \\ \dot{\hat{\gamma}} &= e_1 e_4 + k_7 e_\gamma \\ \dot{\hat{r}} &= e_2(e_1 - y_1 y_3 + x_1 x_3) + k_8 e_r\end{aligned} \tag{14}$$

where $k_5, k_6, k_7$ and $k_8$ are positive constants.

Substituting (14) into (12), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_\alpha^2 - k_6 e_\beta^2 - k_7 e_\gamma^2 - k_8 e_r^2 \tag{15}$$

which is a negative definite function on $R^8$.

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error $e_i, (i = 1, 2, 3, 4)$ and the parameter estimation error $e_\alpha, e_\beta, e_\gamma, e_r$ decay to zero exponentially with time.

Hence, we have proved the following result.

**Theorem 1.** *The identical hyperchaotic Lorenz-Stenflo systems (3) and (4) with unknown parameters are globally and exponentially synchronized by the adaptive control law (7), where the update law for the parameter estimates is given by (14) and* $k_i, (i = 1, 2, \ldots, 8)$ *are positive constants.* ■

### 3.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the 4-D chaotic systems (3) and (4) with the adaptive control law (14) and the parameter update law (14) using MATLAB.

We take $k_i = 2$ for $i = 1, 2, \ldots, 8$.

For the Lorenz-Stenflo systems (3) and (4), the parameter values are taken as

$$\alpha = 2.0, \quad \beta = 0.7, \quad \gamma = 1.5 \quad \text{and} \quad r = 26.0.$$

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0) = 3, \quad \hat{b}(0) = 5, \quad \hat{r}(0) = 8, \quad \hat{d}(0) = 4.$$

The initial values of the master system (3) are taken as

$$x_1(0) = 16, \quad x_2(0) = 12, \quad x_3(0) = 20, \quad x_4(0) = 35.$$

The initial values of the slave system (4) are taken as

$$y_1(0) = 22, \quad y_2(0) = 16, \quad y_3(0) = 15, \quad y_4(0) = 7.$$

Figure 3 depicts the complete synchronization of the identical LS systems (3) and (4). Figure 4 shows that the estimated values of the parameters, viz. $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{r}$ converge to the system parameters $\alpha = 2.0, \quad \beta = 0.7, \quad \gamma = 1.5$ and $r = 26.0.$

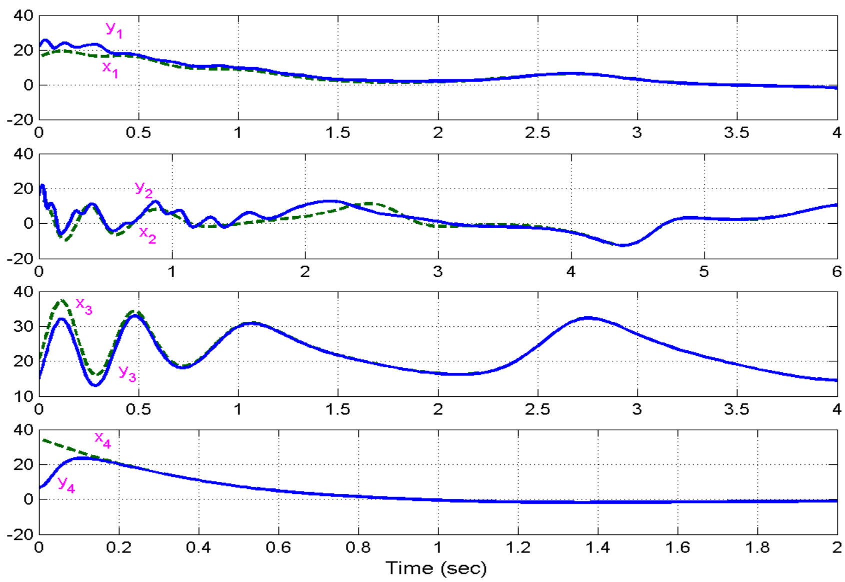


Figure 3. Complete Synchronization of the Lorenz-Stenflo Systems

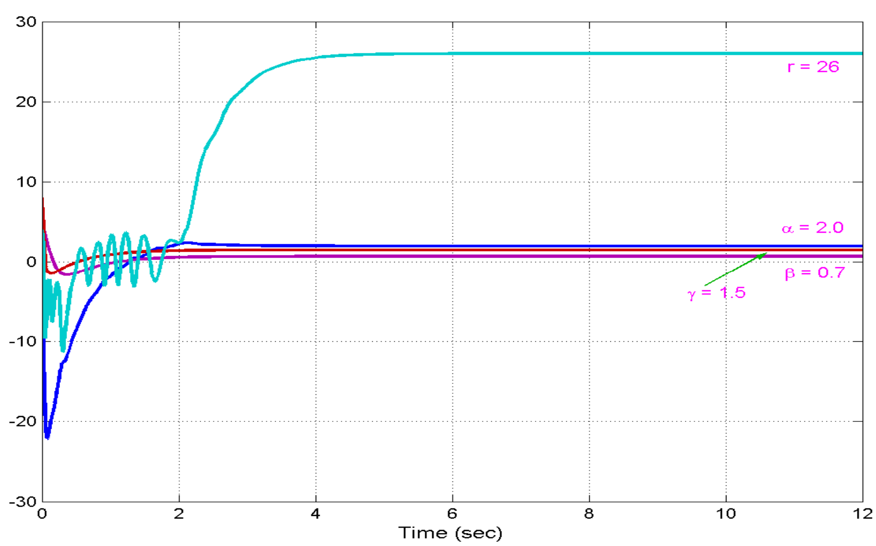


Figure 4. Parameter Estimates $\hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t), \hat{r}(t)$

# 4. ADAPTIVE SYNCHRONIZATION OF IDENTICAL QI SYSTEMS

## 4.1 Theoretical Results

In this section, we discuss the adaptive synchronization of identical Qi systems ([29], 2005), where the parameters of the master and slave systems are unknown.

As the master system, we consider the Qi dynamics described by

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1) + x_2 x_3 x_4 \\ \dot{x}_2 &= b(x_1 + x_2) - x_1 x_3 x_4 \\ \dot{x}_3 &= -c x_3 + x_1 x_2 x_4 \\ \dot{x}_4 &= -d x_4 + x_1 x_2 x_3\end{aligned} \tag{16}$$

where $x_1, x_2, x_3, x_4$ are the states and $a, b, c, d$ are unknown real constant parameters of the system.

As the slave system, we consider the controlled Qi dynamics described by

$$\begin{aligned}\dot{y}_1 &= a(y_2 - y_1) + y_2 y_3 y_4 + u_1 \\ \dot{y}_2 &= b(y_1 + y_2) - y_1 y_3 y_4 + u_2 \\ \dot{y}_3 &= -c y_3 + y_1 y_2 y_4 + u_3 \\ \dot{y}_4 &= -d y_4 + y_1 y_2 y_3 + u_4\end{aligned} \tag{17}$$

where $y_1, y_2, y_3, y_4$ are the states and $u_1, u_2, u_3, u_4$ are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

$$e_i = y_i - x_i, \quad (i = 1, 2, 3, 4) \tag{18}$$

The error dynamics is easily obtained as

$$\begin{aligned}\dot{e}_1 &= a(e_2 - e_1) + y_2 y_3 y_4 - x_2 x_3 x_4 + u_1 \\ \dot{e}_2 &= b(e_1 + e_2) - y_1 y_3 y_4 + x_1 x_3 x_4 + u_2 \\ \dot{e}_3 &= -c e_3 + y_1 y_2 y_4 - x_1 x_2 x_4 + u_3 \\ \dot{e}_4 &= -d e_4 + y_1 y_2 y_3 - x_1 x_2 x_3 + u_4\end{aligned} \tag{19}$$

Let us now define the adaptive control functions

$$\begin{aligned}u_1(t) &= -\hat{a}(e_2 - e_1) - y_2 y_3 y_4 + x_2 x_3 x_4 - k_1 e_1 \\ u_2(t) &= -\hat{b}(e_1 + e_2) + y_1 y_3 y_4 - x_1 x_3 x_4 - k_2 e_2 \\ u_3(t) &= \hat{c} e_3 - y_1 y_2 y_4 + x_1 x_2 x_4 - k_3 e_3 \\ u_4(t) &= \hat{d} e_4 - y_1 y_2 y_3 + x_1 x_2 x_3 - k_4 e_4\end{aligned} \tag{20}$$

where $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are estimates of $a, b, c$ and $d$, respectively, and $k_i, (i = 1, 2, 3, 4)$ are positive constants.

Substituting (20) into (19), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (a - \hat{a})(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= (b - \hat{b})(e_1 + e_2) - k_2 e_2 \\ \dot{e}_3 &= -(c - \hat{c})e_3 - k_3 e_3 \\ \dot{e}_4 &= -(d - \hat{d})e_4 - k_4 e_4 \end{aligned} \tag{21}$$

Let us now define the parameter estimation errors as

$$e_a = a - \hat{a}, \quad e_b = b - \hat{b}, \quad e_c = c - \hat{c} \quad \text{and} \quad e_d = d - \hat{d}. \tag{22}$$

Substituting (22) into (21), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_a(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= e_b(e_1 + e_2) - k_2 e_2 \\ \dot{e}_3 &= -e_c e_3 - k_3 e_3 \\ \dot{e}_4 &= -e_d e_4 - k_4 e_4 \end{aligned} \tag{23}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V(e_1, e_2, e_3, e_4, e_a, e_b, e_c, e_d) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2\right) \tag{24}$$

which is a positive definite function on $R^8$.

We also note that

$$\dot{e}_a = -\dot{\hat{a}}, \quad \dot{e}_b = -\dot{\hat{b}}, \quad \dot{e}_c = -\dot{\hat{c}} \quad \text{and} \quad \dot{e}_d = -\dot{\hat{d}} \tag{25}$$

Differentiating (24) along the trajectories of (23) and using (25), we obtain

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a\left[e_1(e_2 - e_1) - \dot{\hat{a}}\right] \\ &+ e_b\left[e_2(e_1 + e_2) - \dot{\hat{b}}\right] + e_c\left[-e_3^2 - \dot{\hat{c}}\right] + e_d\left[e_4^2 - \dot{\hat{d}}\right] \end{aligned} \tag{26}$$

In view of Eq. (26), the estimated parameters are updated by the following law:

$$\begin{aligned}
\dot{\hat{a}} &= e_1(e_2 - e_1) + k_5 e_a \\
\dot{\hat{b}} &= e_2(e_1 + e_2) + k_6 e_b \\
\dot{\hat{c}} &= -e_3^2 + k_7 e_c \\
\dot{\hat{d}} &= -e_4^2 + k_8 e_d
\end{aligned} \tag{27}$$

where $k_5, k_6, k_7$ and $k_8$ are positive constants.

Substituting (27) into (26), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_a^2 - k_6 e_b^2 - k_7 e_c^2 - k_8 e_d^2 \tag{28}$$

which is a negative definite function on $R^8$.

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error $e_i, (i = 1, 2, 3, 4)$ and the parameter estimation error $e_a, e_b, e_c, e_d$ decay to zero exponentially with time.

Hence, we have proved the following result.

**Theorem 2.** *The identical Qi systems (16) and (17) with unknown parameters are globally and exponentially synchronized by the adaptive control law (20), where the update law for the parameter estimates is given by (27) and* $k_i, (i = 1, 2, \ldots, 8)$ *are positive constants.* ■

## 4.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the chaotic systems (16) and (17) with the adaptive control law (14) and the parameter update law (27) using MATLAB. We take $k_i = 2$ for $i = 1, 2, \ldots, 8$.

For the Qi systems (16) and (17), the parameter values are taken as

$$a = 30, \quad b = 10, \quad c = 1 \quad \text{and} \quad d = 10.$$

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0) = 6, \quad \hat{b}(0) = 2, \quad \hat{c}(0) = 9, \quad \hat{d}(0) = 14$$

The initial values of the master system (16) are taken as

$$x_1(0) = 12, \quad x_2(0) = 26, \quad x_3(0) = 42, \quad x_4(0) = 16.$$

The initial values of the slave system (17) are taken as

$$y_1(0) = 18, \quad y_2(0) = 7, \quad y_3(0) = 15, \quad y_4(0) = 29.$$

Figure 5 depicts the complete synchronization of the identical Qi systems (16) and (17). Figure 6 shows that the estimated values of the parameters, viz. $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ converge to the system parameters $a = 30, \ b = 10, \ c = 1$ and $d = 10.$

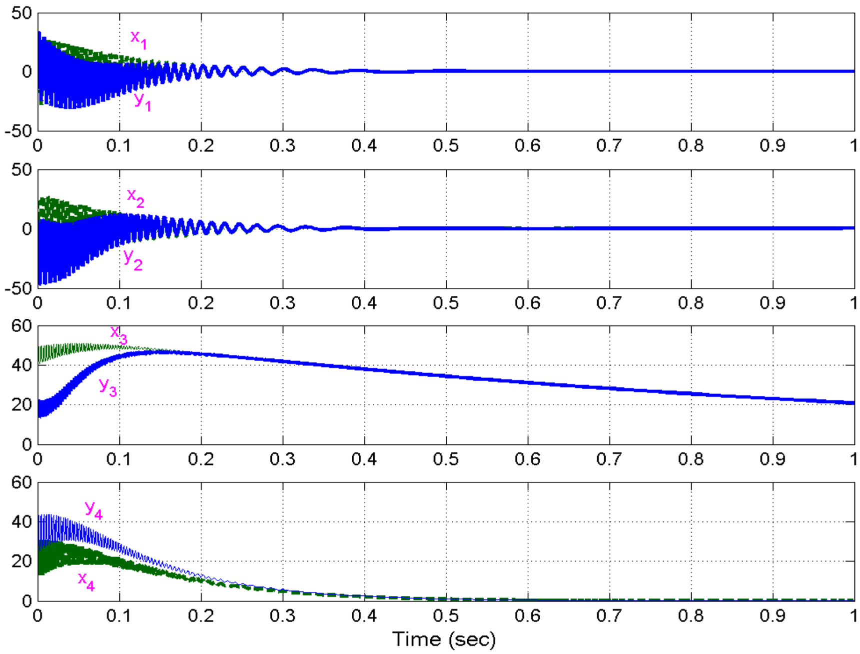


Figure 5. Complete Synchronization of the Qi Systems

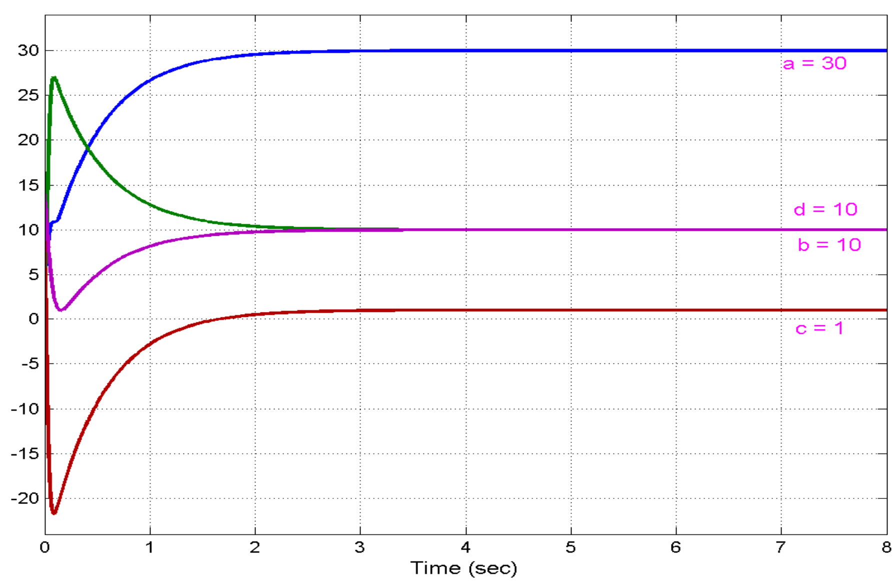


Figure 6. Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t)$

## 5. Adaptive Synchronization of Non-Identical Lorenz-Stenflo and Qi Systems

### 5.1 Theoretical Results

In this section, we discuss the adaptive synchronization of non-identical Lorenz-Stenflo system ([28], 2001) and Qi system ([29], 2005), where the parameters of the master and slave systems are unknown.

As the master system, we consider the LS dynamics described by

$$\begin{aligned}\dot{x}_1 &= \alpha(x_2 - x_1) + \gamma x_4 \\ \dot{x}_2 &= x_1(r - x_3) - x_2 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \\ \dot{x}_4 &= -x_1 - \alpha x_4\end{aligned} \tag{29}$$

where $x_1, x_2, x_3, x_4$ are the states and $\alpha, \beta, \gamma, r$ are unknown real constant parameters of the system.

As the slave system, we consider the controlled Qi dynamics described by

$$\begin{aligned}\dot{y}_1 &= a(y_2 - y_1) + y_2 y_3 y_4 + u_1 \\ \dot{y}_2 &= b(y_1 + y_2) - y_1 y_3 y_4 + u_2 \\ \dot{y}_3 &= -c y_3 + y_1 y_2 y_4 + u_3 \\ \dot{y}_4 &= -d y_4 + y_1 y_2 y_3 + u_4\end{aligned} \tag{30}$$

where $y_1, y_2, y_3, y_4$ are the states, $a, b, c, d$ are unknown real constant parameters of the system and $u_1, u_2, u_3, u_4$ are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

$$e_i = y_i - x_i, \quad (i = 1, 2, 3, 4) \tag{31}$$

The error dynamics is easily obtained as

$$\begin{aligned}\dot{e}_1 &= a(y_2 - y_1) - \alpha(x_2 - x_1) - \gamma x_4 + y_2 y_3 y_4 + u_1 \\ \dot{e}_2 &= b(y_1 + y_2) + x_2 - r x_1 + x_1 x_3 - y_1 y_3 y_4 + u_2 \\ \dot{e}_3 &= -c y_3 + \beta x_3 + y_1 y_2 y_4 - x_1 x_2 + u_3 \\ \dot{e}_4 &= -d y_4 + \alpha x_4 + x_1 + y_1 y_2 y_3 + u_4\end{aligned} \tag{32}$$

Let us now define the adaptive control functions

$$\begin{aligned} u_1(t) &= -\hat{a}(y_2 - y_1) + \hat{\alpha}(x_2 - x_1) + \hat{\gamma} x_4 - y_2 y_3 y_4 - k_1 e_1 \\ u_2(t) &= -\hat{b}(y_1 + y_2) - x_2 + \hat{r} x_1 - x_1 x_3 + y_1 y_3 y_4 - k_2 e_2 \\ u_3(t) &= \hat{c} y_3 - \hat{\beta} x_3 - y_1 y_2 y_4 + x_1 x_2 - k_3 e_3 \\ u_4(t) &= \hat{d} y_4 - \hat{\alpha} x_4 - x_1 - y_1 y_2 y_3 - k_4 e_4 \end{aligned} \tag{33}$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{r}$ are estimates of $a, b, c, d, \alpha, \beta, \gamma$ and $r$, respectively, and $k_i, (i = 1, 2, 3, 4)$ are positive constants.

Substituting (33) into (32), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (a - \hat{a})(y_2 - y_1) - (\alpha - \hat{\alpha})(x_2 - x_1) - (\gamma - \hat{\gamma}) x_4 - k_1 e_1 \\ \dot{e}_2 &= (b - \hat{b})(y_1 + y_2) - (r - \hat{r}) x_1 - k_2 e_2 \\ \dot{e}_3 &= -(c - \hat{c}) y_3 + (\beta - \hat{\beta}) x_3 - k_3 e_3 \\ \dot{e}_4 &= -(d - \hat{d}) y_4 + (\alpha - \hat{\alpha}) x_4 - k_4 e_4 \end{aligned} \tag{34}$$

Let us now define the parameter estimation errors as

$$\begin{aligned} & e_a = a - \hat{a}, \;\; e_b = b - \hat{b}, \;\; e_c = c - \hat{c}, \; e_d = d - \hat{d} \\ & e_\alpha = \alpha - \hat{\alpha}, \;\; e_\beta = \beta - \hat{\beta}, \;\; e_\gamma = \gamma - \hat{\gamma}, \; e_r = r - \hat{r} \end{aligned} \tag{35}$$

Substituting (35) into (32), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_a (y_2 - y_1) - e_\alpha (x_2 - x_1) - e_\gamma x_4 - k_1 e_1 \\ \dot{e}_2 &= e_b (y_1 + y_2) - e_r x_1 - k_2 e_2 \\ \dot{e}_3 &= -e_c y_3 + e_\beta x_3 - k_3 e_3 \\ \dot{e}_4 &= -e_d y_4 + e_\alpha x_4 - k_4 e_4 \end{aligned} \tag{36}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V = \frac{1}{2}\left( e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_r^2 \right) \tag{37}$$

which is a positive definite function on $R^{12}$.

We also note that

$$\dot{e}_a = -\dot{\hat{a}}, \; \dot{e}_b = -\dot{\hat{b}}, \;\; \dot{e}_c = -\dot{\hat{c}}, \; \dot{e}_d = -\dot{\hat{d}}, \; \dot{e}_\alpha = -\dot{\hat{\alpha}}, \;\; \dot{e}_\beta = -\dot{\hat{\beta}}, \;\; \dot{e}_\gamma = -\dot{\hat{\gamma}}, \;\; \dot{e}_r = -\dot{\hat{r}} \tag{38}$$

Differentiating (37) along the trajectories of (36) and using (38), we obtain

$$\begin{aligned}\dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a \left[ e_1 (y_2 - y_1) - \dot{\hat{a}} \right] + e_b \left[ e_2 (y_1 + y_2) - \dot{\hat{b}} \right] \\ &+ e_c \left[ -e_3 y_3 - \dot{\hat{c}} \right] + e_d \left[ -e_4 y_4 - \dot{\hat{d}} \right] + e_\alpha \left[ -e_1 (x_2 - x_1) + e_4 x_4 - \dot{\hat{\alpha}} \right] \\ &+ e_\beta \left[ e_3 x_3 - \dot{\hat{\beta}} \right] + e_\gamma \left[ -e_1 x_4 - \dot{\hat{\gamma}} \right] + e_r \left[ -e_2 x_1 - \dot{\hat{r}} \right]\end{aligned} \tag{39}$$

In view of Eq. (39), the estimated parameters are updated by the following law:

$$\begin{aligned} \dot{\hat{a}} &= e_1 (y_2 - y_1) + k_5 e_a, & \dot{\hat{\alpha}} &= -e_1 (x_2 - x_1) + e_4 x_4 + k_9 e_\alpha \\ \dot{\hat{b}} &= e_2 (y_1 + y_2) + k_6 e_b, & \dot{\hat{\beta}} &= e_3 x_3 + k_{10} e_\beta \\ \dot{\hat{c}} &= -e_3 y_3 + k_7 e_c, & \dot{\hat{\gamma}} &= -e_1 x_4 + k_{11} e_\gamma \\ \dot{\hat{d}} &= -e_4 y_4 + k_8 e_d, & \dot{\hat{r}} &= -e_2 x_1 + k_{12} e_r \end{aligned} \tag{40}$$

where $k_5, k_6, k_7, k_8, k_9, k_{10}, k_{11}$ and $k_{12}$ are positive constants.

Substituting (40) into (39), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_a^2 - k_6 e_b^2 - k_7 e_c^2 - k_8 e_d^2 - k_9 e_\alpha^2 - k_{10} e_\beta^2 - k_{11} e_\gamma^2 - k_{12} e_r^2 \tag{41}$$

which is a negative definite function on $R^{12}$.

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error $e_i, (i = 1, 2, 3, 4)$ and the parameter estimation error decay to zero exponentially with time.

Hence, we have proved the following result.

**Theorem 3.** *The non-identical Lorenz-Stenflo system (29) and Qi system (30) with unknown parameters are globally and exponentially synchronized by the adaptive control law (33), where the update law for the parameter estimates is given by (40) and* $k_i, (i = 1, 2, \ldots, 12)$ *are positive constants.* ■

## 5.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the chaotic systems (29) and (30) with the adaptive control law (27) and the parameter update law (40) using MATLAB. We take $k_i = 2$ for $i = 1, 2, \ldots, 12$.

For the Lorenz-Stenflo system (29) and Qi system (30), the parameter values are taken as

$$a = 30, \ \ b = 10, \ \ c = 1, \ \ d = 10, \ \ \alpha = 2.0, \ \ \beta = 0.7, \ \ \gamma = 1.5, \ \ r = 26. \tag{42}$$

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0)=5,\ \hat{b}(0)=10,\ \hat{c}(0)=2,\ \hat{d}(0)=4,\ \hat{\alpha}(0)=6,\ \hat{\beta}(0)=9,\ \hat{\gamma}(0)=7,\ \hat{r}(0)=5$$

The initial values of the master system (29) are taken as

$$x_1(0)=32,\ \ x_2(0)=14,\ \ x_3(0)=25,\ \ x_4(0)=27.$$

The initial values of the slave system (30) are taken as

$$y_1(0)=14,\ \ y_2(0)=17,\ \ y_3(0)=30,\ \ y_4(0)=20.$$

Figure 7 depicts the complete synchronization of the non-identical Lorenz-Stenflo and Qi systems. Figure 8 shows that the estimated values of the parameters, viz. $\hat{a},\hat{b},\hat{c},\hat{d},\hat{\alpha},\hat{\beta},\hat{\gamma}$ and $\hat{r}$ converge to the original values of the parameters given in (42).

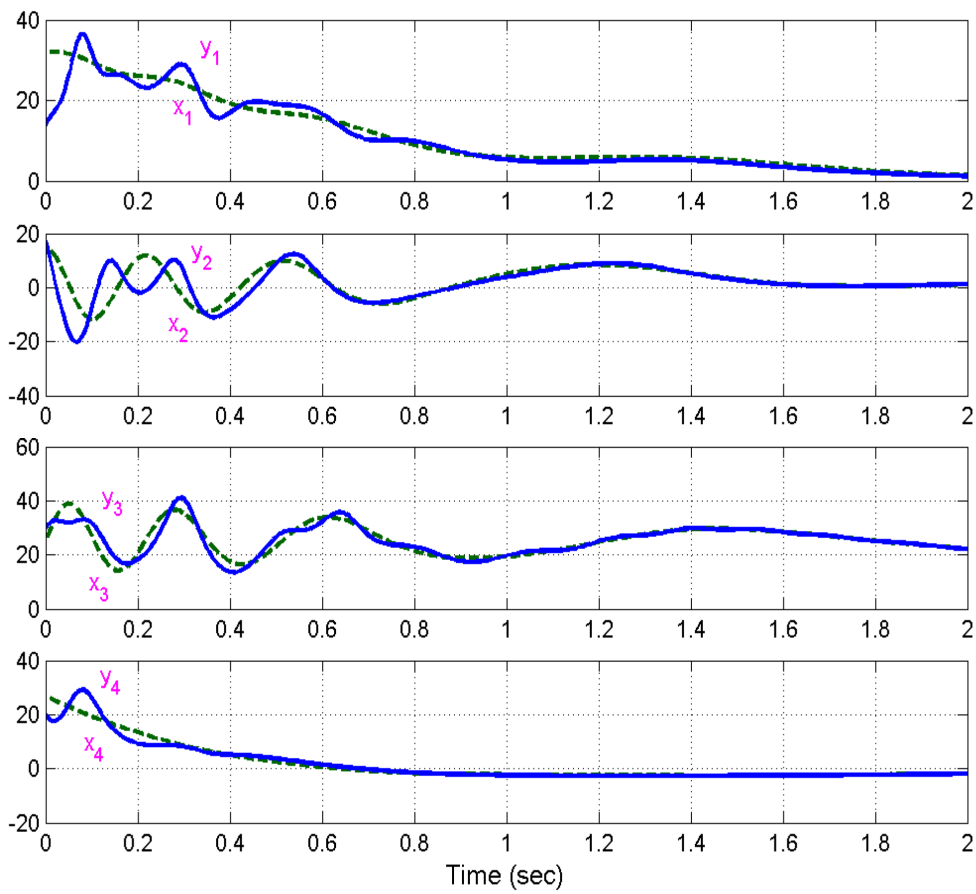


Figure 7. Complete Synchronization of Lorenz-Stenflo and Qi Systems

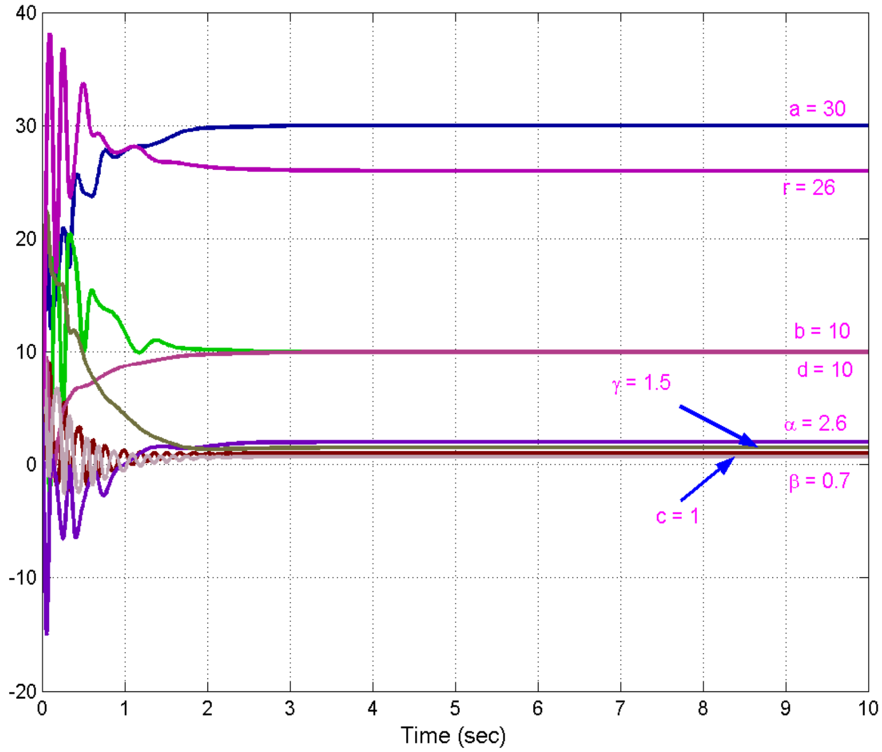


Figure 8. Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t), \hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t), \hat{r}(t)$

## 5. CONCLUSIONS

In this paper, we have applied adaptive control method for the global chaos synchronization of identical Lorenz-Stenflo systems (2001), identical Qi systems (2005) and non-identical LS and Qi systems with unknown parameters. The adaptive synchronization results derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is a very effective and convenient for achieving chaos synchronization for the uncertain hyperchaotic systems discussed in this paper. Numerical simulations are shown to demonstrate the effectiveness of the adaptive synchronization schemes derived in this paper for the synchronization of identical and non-identical uncertain LS and Qi systems.

## REFERENCES

[1] Alligood, K.T., Sauer, T. & Yorke, J.A. (1997) *Chaos: An Introduction to Dynamical Systems*, Springer, New York.

[2] Vincent, U.E., Njah, A.N., Akinlade, O. & Solarin, A.R.T. (2004) "Phase synchronization in unidirectionally coupled chaotic ratchets," *Chaos*, Vol. 14, pp 1018-1025.

[3] Vincent, U.E., Njah, A.N., Akinlade, O. & Solarin, A.R.T. (2006) "Phase synchronization in bidirectionally coupled chaotic ratchets," *Physica A*, Vol. 360, pp 180-196.

[4] Lakshmanan, M. & Murali, K. (1996) *Nonlinear Oscillators: Controlling and Synchronization*, World Scientific, Singapore.

[5] Han, S.K., Kerrer, C. & Kuramoto, Y. (1995) "Dephasing and bursting in coupled neural oscillators", *Phys. Rev. Lett.,* Vol. 75, pp 3190-3193.

[6] Blasius, B., Huppert, A. & Stone, L. (1999) "Complex dynamics and phase synchronization in spatially extended ecological system", *Nature*, Vol. 399, pp 354-359.

[7] Niu, H., Zhang, Q. & Zhang, Y. (2002) "The chaos synchronization of a singular chemical model and a Williamowski-Rossler model," *International Journal of Information and Systems Sciences*, Vol. 6, No. 4, pp 355-364.

[8] Kocarev, L. & Parlitz, U. (1995) "General approach for chaotic synchronization with applications to communication," *Physical Review Letters*, Vol. 74, pp 5028-5030.

[9] Boccaletti, S., Farini, A. & Arecchi, F.T. (1997) "Adaptive synchronization of chaos for secure communication," Physical Review E, Vol. 55, No. 5,

[10] Tao, Y. (1999) "Chaotic secure communication systems – history and new results," *Telecommun. Review*, Vol. 9, pp 597-634.

[11] Pecora, L.M. & Carroll, T.L. (1990) "Synchronization in chaotic systems", *Phys. Rev. Lett.*, Vol. 64, pp 821-824.

[12] Ott, E., Grebogi, C. & Yorke, J.A. (1990) "Controlling chaos", *Phys. Rev. Lett.*, Vol. 64, pp 1196-1199.

[13] Ho, M.C. & Hung, Y.C. (2002) "Synchronization of two different chaotic systems using generalized active network," *Physics Letters A,* Vol. 301, pp 424-428.

[14] Huang, L., Feng, R. & Wang, M. (2005) "Synchronization of chaotic systems via nonlinear control," *Physical Letters A*, Vol. 320, pp 271-275.

[15] Chen, H.K. (2005) "Global chaos synchronization of new chaotic systems via nonlinear control," *Chaos, Solitons & Fractals*, Vol. 23, pp 1245-1251.

[16] Sundarapandian, V. & Suresh, R. (2011) "Global chaos synchronization of hyperchaotic Qi and Jia systems by nonlinear control," *International Journal of Distributed and Parallel Systems*, Vol. 2, No. 2, pp. 83-94.

[17] Sundarapandian, V. (2011) "Hybrid chaos synchronization of hyperchaotic Liu and hyperchaotic Chen systems by active nonlinear control," *International Journal of Computer Science, Engineering and Information Technology*, Vol. 1, No. 2, pp. 1-14.

[18] Lu, J., Wu, X., Han, X. & Lü, J. (2004) "Adaptive feedback synchronization of a unified chaotic system," *Physics Letters A*, Vol. 329, pp 327-333.

[19] Sundarapandian, V. (2011) "Adaptive control and synchronization of hyperchaotic Liu system," *International Journal of Computer Science, Engineering and Information Technology*, Vol. 1, No. 2, pp. 29-40.

[20] Sundarapandian, V. (2011) "Adaptive control and synchronization of hyperchaotic Newton-Leipnik system," *International Journal of Advanced Information Technology*, Vol. 1, No. 3, pp. 22-33.

[21] Park, J.H. & Kwon, O.M. (2003) "A novel criterion for delayed feedback control of time-delay chaotic systems," *Chaos, Solitons & Fractals*, Vol. 17, pp 709-716.

[22] Wu, X. & Lü, J. (2003) "Parameter identification and backstepping control of uncertain Lü system," *Chaos, Solitons & Fractals*, Vol. 18, pp 721-729.

[23] Tan, X., Zhang, J. & Yang, Y. (2003) "Synchronizing chaotic systems using backstepping design," *Chaos, Solitons & Fractals*, Vol. 16, pp. 37-45.

[24] Zhao, J. & Lu, J. (2006) "Using sampled-data feedback control and linear feedback synchronization in a new hyperchaotic system," *Chaos, Solitons and Fractals*, Vol. 35, pp 376-382.

[25] Slotine, J.E. & Sastry, S.S. (1983) "Tracking control of nonlinear systems using sliding surface with application to robotic manipulators," *Internat. J. Control*, Vol. 38, pp 465-492.

[26] Utkin, V.I. (1993) "Sliding mode control design principles and applications to electric drives," *IEEE Trans. Industrial Electronics*, Vol. 40, pp 23-36, 1993.

[27] Sundarapandian, V. (2011) "Sliding mode control based global chaos synchronization of Sprott F systems," *International Journal of Advances in Science and Technology*, Vol. 2, No. 5, pp 1-10.

[28] Stenflo, L. (2001) "Generalized Lorenz equation for acoustic waves in the atmosphere," *Physica Scripta*, Vol. 53, No. 1, pp 177-180.

[29] Qi, G., Chen, Z., Du, S. & Yuan, Z. (2005) "On a four-dimensional chaotic system," *Chaos, Solitons and Fractals,* Vol. 23, pp 1671-1682.

[30] Hahn, W. (1967) *The Stability of Motion*, Springer, New York.


## Author

**Dr. V. Sundarapandian** is a Professor (Systems & Control Engineering) in the R & D Centre at Vel Tech Dr. RR & Dr. SR Technical University, Chennai, India. His current research areas are Linear and Nonlinear Control Systems, Chaos Theory, Dynamical Systems, Soft Computing, etc. He has published over 150 refereed international journal publications. He has published over 45 papers in International Conferences and over 90 papers in National Conferences. He has published two text-books titled *Numerical Linear Algebra* and *Probability, Statistics and Queueing Theory* with Prentice Hall of India, New Delhi, India. He is a Senior Member of AIRCC and the Editor-in-Chief of the AIRCC international journals – *International Journal of Instrumentation and Control Systems* and *International Journal of Control Theory and Computer Modeling*. He is the Associate Editor of many Control Journals like *International Journal of Control Theory and Applications*, *International Journal of Computer Information and Sciences,* and *International Journal of Advances in Science and Technology*. He has delivered many Key Note Lectures on Nonlinear Control Systems, Modern Control Systems, Chaos and Control, Mathematical Modelling, Scientific Computing with SCILAB/MATLAB, etc.