

STATE FEEDBACK CONTROLLER DESIGN FOR THE OUTPUT REGULATION OF SPROTT-H SYSTEM

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ABSTRACT

This paper investigates the problem of state feedback controller design for the output regulation of Sprott-K chaotic system, which is one of the simple, classical, three-dimensional chaotic systems discovered by J.C. Sprott (1994). Explicitly, we have derived new state feedback control laws for the constant tracking problem for the Sprott-H system. The state feedback control laws have been derived using the regulator equations of C.I. Byrnes and A. Isidori (1990). The output regulation of the Sprott H chaotic system has important applications in Electronics and Communication Engineering. Numerical simulations are shown to illustrate the effectiveness of the control schemes proposed in this paper for the output regulation of the Sprott-H chaotic system.

KEYWORDS

Chaos; feedback control; Sprott-H system; nonlinear control systems; output regulation.

1. INTRODUCTION

Regulating the output of a nonlinear plant is one of the fundamental problems in systems theory. Basically, the output regulation problem is to control a fixed linear or nonlinear plant in order to have its output tracking reference signals produced by some external generator (the *exosystem*). For linear control systems, the output regulation problem was solved by Francis and Wonham ([1], 1975). For nonlinear control systems, the output regulation problem was solved by Byrnes and Isidori ([2], 1990) generalizing the internal model principle obtained by Francis and Wonham [1]. Using Centre Manifold Theory [3], Byrnes and Isidori derived *regulator equations*, which characterize the solution of the output regulation problem of nonlinear control systems satisfying some stability assumptions.

The output regulation problem for nonlinear control systems has been studied intensively and extensively in the last two decades [4-14]. In [4], Mahmoud and Khalil derived results on the asymptotic regulation of minimum phase nonlinear systems using output feedback. In [5], Fridman derived output regulation results for nonlinear control systems with delay. In [6-7], Chen and Huang derived results on the robust output regulation for output feedback systems with nonlinear exosystems. In [8], Liu and Huang derived results on the global robust output regulation problem for lower triangular nonlinear systems with unknown control direction.

In [9], Immonen derived results on the practical output regulation for bounded linear infinite-dimensional state space systems. In [10], Pavlov, Van de Wouw and Nijmeijer derived results on the global nonlinear output regulation using convergence-based controller design. In [11], Xi and Dong derived results on the global adaptive output regulation of a class of nonlinear systems with nonlinear exosystems. In [12-14], Serrani, Isidori and Marconi obtained results on the semi-

global and global output regulation problem for minimum-phase nonlinear systems. In [15-16], we derived output regulation results for some three-dimensional chaotic systems using regulator equations.

In this paper, we solve the output regulation problem for the Sprott I chaotic system ([17], 1994). We derive state feedback control laws solving the constant regulation problem of the Sprott-H chaotic system using the regulator equations of Byrnes and Isidori (1990). The Sprott-H chaotic system is a classical and simple three-dimensional chaotic system proposed by J.C. Sprott (1994). It has important applications in Electronics and Communication Engineering.

This paper is organized as follows. In Section 2, we review the output regulation problem for nonlinear control systems and the regulator equations of Byrnes and Isidori (1990), which provide a solution to the output regulation problem under some stability assumptions. In Section 3, we detail the feedback control design for the output regulation problem for the Sprott H chaotic system for the important case of constant reference signals (*set-point signals*). Section 4 describes the numerical simulations for the constant regulation problem of the Sprott H chaotic system. Section 5 contains a summary of the main results derived in this paper.

2. REVIEW OF THE OUTPUT REGULATION PROBLEM FOR NONLINEAR CONTROL SYSTEMS

In this section, we consider a multi-variable nonlinear control system described by

$$\dot{x} = f(x) + g(x)u + p(x)\omega \quad (1a)$$

$$\dot{\omega} = s(\omega) \quad (1b)$$

$$e = h(x) - q(\omega) \quad (2)$$

Here, the differential equation (1a) describes the *plant dynamics* with state x defined in a neighbourhood X of the origin of R^n and the input u takes values in R^m subject to the effect of a disturbance represented by the vector field $p(x)\omega$. The differential equation (1b) describes an autonomous system, known as the *exosystem*, defined in a neighbourhood W of the origin of R^k , which models the class of disturbance and reference signals taken into consideration. The equation (2) defines the error between the actual plant output $h(x) \in R^p$ and a reference signal $q(\omega)$, which models the class of disturbance and reference signals taken into consideration.

We also assume that all the constituent mappings of the system (1) and the error equation (2), namely, f, g, p, s, h and q are continuously differentiable mappings vanishing at the origin, i.e.

$$f(0) = 0, g(0) = 0, p(0) = 0, s(0) = 0, h(0) = 0 \text{ and } q(0) = 0.$$

Thus, for $u = 0$, the composite system (1) has an equilibrium $(x, \omega) = (0, 0)$ with zero error (2).

A state feedback controller for the composite system (1) has the form

$$u = \rho(x, \omega) \quad (3)$$

where ρ is a continuously differentiable mapping defined on $X \times W$ such that $\rho(0, 0) = 0$.

Upon substitution of the feedback control law (3) into (1), we get the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\rho(x, \omega) + p(x)\omega \\ \dot{\omega} &= s(\omega) \end{aligned} \quad (4)$$

The purpose of designing the state feedback controller (3) is to achieve both *internal stability* and *output regulation* of the given nonlinear control system (1). Formally, we can summarize these requirements as follows.

State Feedback Regulator Problem [2]:

Find, if possible, a state feedback control law $u = \rho(x, \omega)$ such that the following conditions are satisfied.

(OR1) [*Internal Stability*] The equilibrium $x = 0$ of the dynamics

$$\dot{x} = f(x) + g(x)\rho(x, 0)$$

is locally exponentially stable.

(OR2) [*Output Regulation*] There exists a neighbourhood $U \subset X \times W$ of $(x, \omega) = (0, 0)$ such that for each initial condition $(x(0), \omega(0)) \in U$, the solution $(x(t), \omega(t))$ of the closed-loop system (4) satisfies

$$\lim_{t \rightarrow \infty} [h(x(t)) - q(\omega(t))] = 0. \quad \blacksquare$$

Byrnes and Isidori [2] solved the output regulation problem stated above under the following two assumptions.

(H1) The exosystem dynamics $\dot{\omega} = s(\omega)$ is neutrally stable at $\omega = 0$, i.e. the exosystem is Lyapunov stable in both forward and backward time at $\omega = 0$.

(H2) The pair $(f(x), g(x))$ has a stabilizable linear approximation at $x = 0$, i.e. if

$$A = \left[\frac{\partial f}{\partial x} \right]_{x=0} \quad \text{and} \quad B = \left[\frac{\partial g}{\partial x} \right]_{x=0},$$

then (A, B) is stabilizable. \blacksquare

Next, we recall the solution of the output regulation problem derived by Byrnes and Isidori [2].

Theorem 1. [2] *Under the hypotheses (H1) and (H2), the state feedback regulator problem is solvable if and only if there exist continuously differentiable mappings $x = \pi(\omega)$ with $\pi(0) = 0$ and $u = \varphi(\omega)$ with $\varphi(0) = 0$, both defined in a neighbourhood of $W^0 \subset W$ of $\omega = 0$ such that the following equations (called the **regulator equations**) are satisfied:*

$$(1) \quad \frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega)) + g(\pi(\omega))\varphi(\omega) + p(\pi(\omega))\omega$$

$$(2) \quad h(\pi(\omega)) - q(\omega) = 0$$

When the regulator equations (1) and (2) are satisfied, a control law solving the state feedback regulator problem is given by

$$u = \varphi(\omega) + K [x - \pi(\omega)]$$

where K is any gain matrix such that $A + BK$ is Hurwitz. \blacksquare

3. OUTPUT REGULATION OF THE SPROTT-H CHAOTIC SYSTEM

In this section, we solve the output regulation problem for the Sprott-H chaotic system ([17], 1994), which is one of the paradigms of the three-dimensional chaotic systems described by the dynamics

$$\begin{aligned} \dot{x}_1 &= -ax_2 + x_3^2 + u \\ \dot{x}_2 &= x_1 + bx_2 \\ \dot{x}_3 &= x_1 - x_3 \end{aligned} \tag{5}$$

where x_1, x_2, x_3 are the states of the system, a, b are positive constant parameters of the system and u is the scalar control.

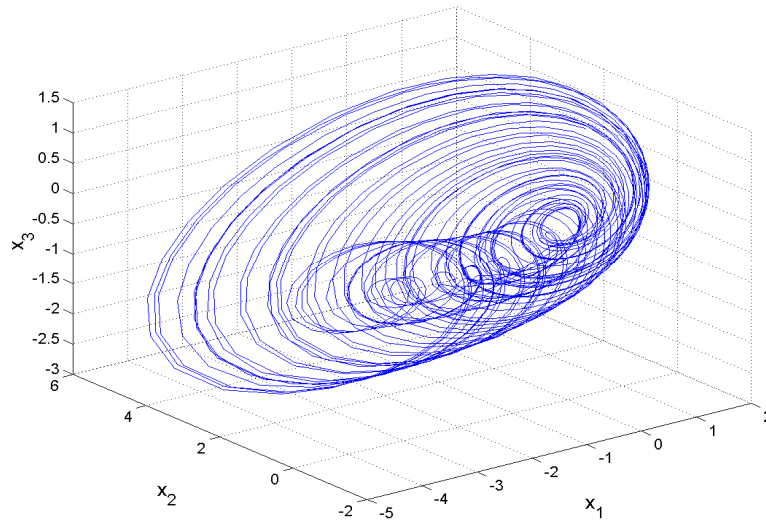


Figure 1. State Orbits of the Sprott-H Chaotic System

J.C. Sprott ([17], 1994) showed that the system (5) has chaotic behaviour when $a = 1, b = 0.5$ and $u = 0$. The chaotic state portrait of (5) is illustrated in Figure 1.

In this paper, we consider the output regulation problem for the tracking of constant reference signals (*set-point signals*).

In this case, the exosystem is given by the scalar dynamics

$$\dot{w} = 0 \tag{6}$$

We note that the assumption (H1) of Theorem 1 holds trivially.

Linearizing the dynamics of the Sprott-H system (5) at $x = 0$, we obtain

$$A = \begin{bmatrix} 0 & -a & 0 \\ 1 & b & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Using Kalman's rank test for controllability ([18], p738), it can be easily seen that the pair (A, B) is completely controllable whenever $b \neq 1$. In the chaotic case, $b = 0.5$. Thus, it is immediate that the linear pair (A, B) is almost always completely controllable including the chaotic case. In particular, using Ackermann's formula, we can find a gain matrix K such that $A + BK$ is Hurwitz with any desired set of stable eigenvalues.

Thus, the assumption (H2) of Theorem 1 also holds.

Hence, Theorem 1 can be applied to solve the constant regulation problem for the Sprott-H system (5).

3.1 The Constant Tracking Problem for x_1

Here, the tracking problem for the Sprott-H chaotic system (5) is given by

$$\begin{aligned}\dot{x}_1 &= -ax_2 + x_3^2 + u \\ \dot{x}_2 &= x_1 + bx_2 \\ \dot{x}_3 &= x_1 - x_3 \\ e &= x_1 - \omega\end{aligned}\tag{7}$$

By Theorem 1, the regulator equations of the system (7) are obtained as

$$\begin{aligned}-a\pi_2(\omega) + \pi_3^2(\omega) + \varphi(\omega) &= 0 \\ \pi_1(\omega) + b\pi_2(\omega) &= 0 \\ \pi_1(\omega) - \pi_3(\omega) &= 0 \\ \pi_1(\omega) - \omega &= 0\end{aligned}\tag{8}$$

Solving the regulator equations (8) for the system (7), we obtain the unique solution as

$$\pi_1(\omega) = \omega, \quad \pi_2(\omega) = -\frac{\omega}{b}, \quad \pi_3(\omega) = \omega \quad \text{and} \quad \varphi(\omega) = -\frac{a\omega}{b} - \omega^2\tag{9}$$

Using Theorem 1 and the solution (9) of the regulator equations for the system (7), we obtain the following result which provides a solution of the output regulation problem for (7).

Theorem 2. *A state feedback control law solving the output regulation problem for the Sprott-H chaotic system (7) is given by*

$$u = \varphi(\omega) + K[x - \pi(\omega)],\tag{10}$$

where $\varphi(\omega)$, $\pi(\omega)$ are defined as in (9) and K is a gain matrix such that $A + BK$ is Hurwitz.

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3.2 The constant Tracking Problem for x_2

Here, the tracking problem for the Sprott-H chaotic system (5) is given by

$$\begin{aligned}\dot{x}_1 &= -ax_2 + x_3^2 + u \\ \dot{x}_2 &= x_1 + bx_2 \\ \dot{x}_3 &= x_1 - x_3 \\ e &= x_2 - \omega\end{aligned}\tag{11}$$

By Theorem 1, the regulator equations of the system (11) are obtained as

$$\begin{aligned}
 -a\pi_2(\omega) + \pi_3^2(\omega) + \varphi(\omega) &= 0 \\
 \pi_1(\omega) + b\pi_2(\omega) &= 0 \\
 \pi_1(\omega) - \pi_3(\omega) &= 0 \\
 \pi_2(\omega) - \omega &= 0
 \end{aligned} \tag{12}$$

Solving the regulator equations (12) for the system (11), we obtain the unique solution as

$$\pi_1(\omega) = -b\omega, \quad \pi_2(\omega) = \omega, \quad \pi_3(\omega) = -b\omega \quad \text{and} \quad \varphi(\omega) = a\omega - b^2\omega^2 \tag{13}$$

Using Theorem 1 and the solution (13) of the regulator equations for the system (11), we obtain the following result which provides a solution of the output regulation problem for (11).

Theorem 3. *A state feedback control law solving the output regulation problem for the Sprott-H chaotic system (11) is given by*

$$u = \varphi(\omega) + K[x - \pi(\omega)], \tag{14}$$

where $\varphi(\omega)$, $\pi(\omega)$ are defined as in (13) and K is a gain matrix such that $A + BK$ is Hurwitz.

■

3.3 The Constant Tracking Problem for x_3

Here, the tracking problem for the Sprott-H chaotic system (5) is given by

$$\begin{aligned}
 \dot{x}_1 &= -ax_2 + x_3^2 + u \\
 \dot{x}_2 &= x_1 + bx_2 \\
 \dot{x}_3 &= x_1 - x_3 \\
 e &= x_3 - \omega
 \end{aligned} \tag{15}$$

By Theorem 1, the regulator equations of the system (15) are obtained as

$$\begin{aligned}
 -a\pi_2(\omega) + \pi_3^2(\omega) + \varphi(\omega) &= 0 \\
 \pi_1(\omega) + b\pi_2(\omega) &= 0 \\
 \pi_1(\omega) - \pi_3(\omega) &= 0 \\
 \pi_3(\omega) - \omega &= 0
 \end{aligned} \tag{16}$$

Solving the regulator equations (16) for the system (15), we obtain the unique solution as

$$\pi_1(\omega) = \omega, \quad \pi_2(\omega) = -\frac{\omega}{b}, \quad \pi_3(\omega) = \omega \quad \text{and} \quad \varphi(\omega) = -\frac{a\omega}{b} - \omega^2 \tag{17}$$

Using Theorem 1 and the solution (17) of the regulator equations for the system (15), we obtain the following result which provides a solution of the output regulation problem for (15).

Theorem 4. *A state feedback control law solving the output regulation problem for the Sprott-H chaotic system (15) is given by*

$$u = \varphi(\omega) + K[x - \pi(\omega)], \tag{18}$$

where $\varphi(\omega)$, $\pi(\omega)$ are defined as in (17) and K is a gain matrix such that $A + BK$ is Hurwitz.

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4. NUMERICAL SIMULATIONS

For simulation, the parameters are chosen as the chaotic case of the Sprott-H system, *viz.*

$$a = 1 \text{ and } b = 0.5.$$

For achieving internal stability of the state feedback regulator problem, we choose a gain matrix K so that

$$A + BK$$

is Hurwitz.

Suppose that we choose a gain matrix K so that $A + BK$ is Hurwitz with stable eigenvalues

$$\{-4, -4, -4\}.$$

Using Ackermann's formula (MATLAB), we obtain

$$K = [-11.5 \quad -59.75 \quad 18.0].$$

For the numerical simulations, the fourth order Runge-Kutta method with step-size $h = 10^{-6}$ is deployed to solve the systems of differential equations (Sprott-H systems) using MATLAB.

4.1 Constant Tracking Problem for x_1

Here, the initial conditions are taken as

$$x_1(0) = 24, \quad x_2(0) = 18, \quad x_3(0) = 22$$

and

$$\omega = 2.$$

The simulation graph is depicted in Figure 2 from which it is clear that the state trajectory $x_1(t)$ tracks the constant reference signal $\omega = 2$ in 3 seconds.

4.2 Constant Tracking Problem for x_2

Here, the initial conditions are taken as

$$x_1(0) = 17, \quad x_2(0) = 19, \quad x_3(0) = 22$$

and

$$\omega = 2.$$

The simulation graph is depicted in Figure 3 from which it is clear that the state trajectory $x_2(t)$ tracks the constant reference signal $\omega = 2$ in 3 seconds.

4.3 Constant Tracking Problem for x_3

Here, the initial conditions are taken as

$$x_1(0) = 15, \quad x_2(0) = 8, \quad x_3(0) = 12$$

and

$$\omega = 2.$$

The simulation graph is depicted in Figure 4 from which it is clear that the state trajectory $x_3(t)$ tracks the constant reference signal $\omega = 2$ in 11 seconds.

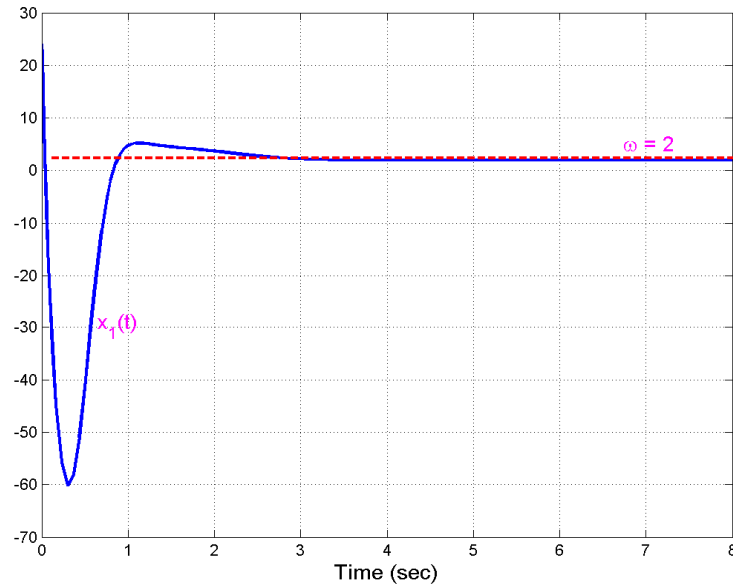


Figure 2. Constant Tracking Problem for x_1

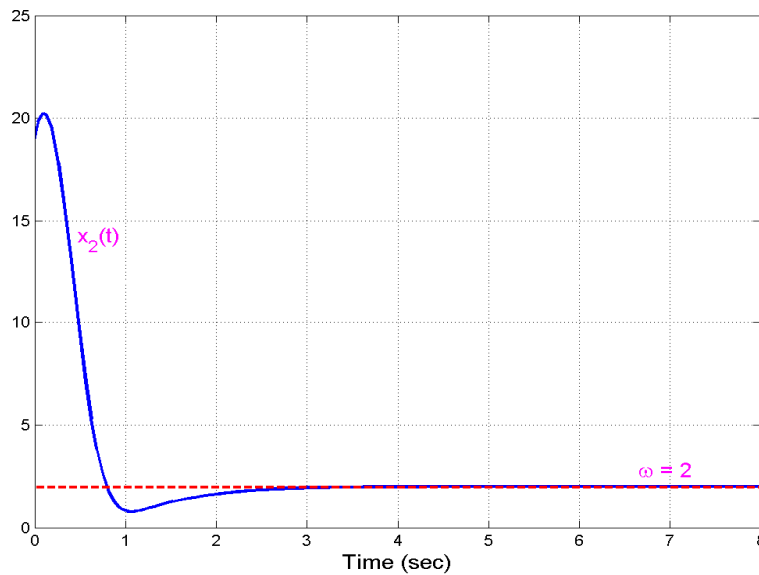
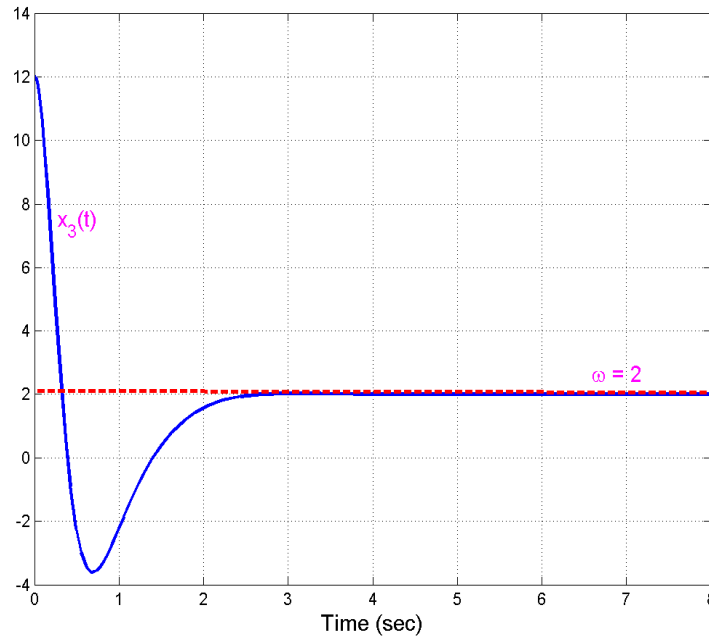


Figure 3. Constant Tracking Problem for x_2

Figure 4. Constant Tracking Problem for x_3

5. CONCLUSIONS

In this paper, we have derive state feedback controllers for the output regulation problem of the Sprott-H chaotic system (1994). The output regulation problem considered for the Sprott-H chaotic system was for the tracking of constant reference signals (*set-point signals*). The state feedback control laws achieving output regulation proposed in this paper were derived using the regulator equations of Byrnes and Isidori (1990). Numerical simulation results were presented in detail to illustrate the effectiveness of the proposed control schemes for the output regulation problem of Sprott-H chaotic system to track constant reference signals.

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