SLIDING MODE CONTROLLER DESIGN FOR GLOBAL CHAOS SYNCHRONIZATION OF HYPERCHAOTIC LORENZ SYSTEMS

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ABSTRACT

This paper derives new results for the design of sliding mode controller for the global chaos synchronization of identical hyperchaotic Lorenz systems (Jia, 2007). The synchronizer results derived in this paper for the complete chaos synchronization of identical hyperchaotic systems are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the sliding mode control method is very effective and convenient to achieve global chaos synchronization of the identical hyperchaotic Lorenz systems. Numerical simulations are shown to illustrate and validate the synchronization schemes derived in this paper for the identical hyperchaotic Lorenz systems.

KEYWORDS

Sliding Mode Control, Hyperchaos, Chaos Synchronization, Hyperchaotic Systems, Hyperchaotic Lorenz System.

1. INTRODUCTION

Chaotic systems are dynamical systems that are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the *butterfly effect* [1]. Chaos is an interesting nonlinear phenomenon and has been extensively studied in the last three decades.
A *hyperchaotic system* is usually characterized as a chaotic system with more than one positive Lyapunov exponent implying that the dynamics expand in more than one direction giving rise to "thicker" and "more complex" chaotic dynamics. The first hyperchaotic system was discovered by Rössler in 1979 [2]. In the last two decades, hyperchaotic systems found many applications in areas such as secure communications, data encryptions, etc.

In most of the chaos synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

Since the pioneering work by Pecora and Carroll ([3], 1990), chaos synchronization problem has been studied extensively and intensively in the literature [3-35]. Chaos theory has been applied to a variety of fields such as physical systems [4], chemical systems [5], ecological systems [6], secure communications [7-9], etc.

In the last two decades, various schemes have been successfully applied for chaos synchronization such as PC method [3], OGY method [10], active control method [11-18], adaptive control method [19-27], time-delay feedback method [28-29], backstepping design method [30], sampled-data feedback method [31], etc.

In this paper, we derive new results based on the sliding mode control [32-35] for the global chaos synchronization of identical hyperchaotic Lorenz systems ([36], Zia, 2007). In robust control systems, the sliding mode control method is often adopted due to its inherent advantages of easy realization, fast response and good transient performance as well as its insensitivity to parameter uncertainties and external disturbances.

This paper has been organized as follows. In Section 2, we describe the problem statement and our methodology using sliding mode control (SMC). In Section 3, we discuss the global chaos synchronization of identical hyperchaotic Lorenz systems. In Section 4, we summarize the main results obtained in this paper.

2. PROBLEM STATEMENT AND OUR METHODOLOGY USING SMC

In this section, we describe the problem statement for the global chaos synchronization for identical chaotic systems and our methodology using sliding mode control (SMC).
Consider the chaotic system described by

$$\dot{x} = Ax + f(x) \tag{1}$$

where $x \in R^n$ is the state of the system, A is the $n \times n$ matrix of the system parameters and $f : R^n \to R^n$ is the nonlinear part of the system.

We consider the system (1) as the *master* or *drive* system.

As the *slave* or *response* system, we consider the following chaotic system described by the dynamics

$$\dot{y} = Ay + f(y) + u \tag{2}$$

where $y \in R^n$ is the state of the system and $u \in R^m$ is the controller to be designed.

If we define the *synchronization error* as

$$e = y - x, \tag{3}$$

then the error dynamics is obtained as

$$\dot{e} = Ae + \eta(x, y) + u, \tag{4}$$

where

$$\eta(x, y) = f(y) - f(x) \tag{5}$$

The objective of the global chaos synchronization problem is to find a controller u such that

$$\lim_{t\to\infty} \|e(t)\| = 0 \quad \text{for all } e(0) \in R^n.$$

To solve this problem, we first define the control u as

$$u = -\eta(x, y) + Bv \tag{6}$$

where B is a constant gain vector selected such that (A, B) is controllable.

Substituting (5) into (4), the error dynamics simplifies to

$$\dot{e} = Ae + Bv \tag{7}$$

which is a linear time-invariant control system with single input v.

Thus, the original global chaos synchronization problem can be replaced by an equivalent problem of stabilizing the zero solution $e = 0$ of the system (7) by a suitable choice of the sliding mode control. In the sliding mode control, we define the variable

$$s(e) = Ce = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n \tag{8}$$

where $C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$ is a constant row vector to be determined.

In the sliding mode control, we constrain the motion of the system (7) to the sliding manifold defined by

$$S = \left\{ x \in R^n \mid s(e) = 0 \right\}$$

which is required to be invariant under the flow of the error dynamics (7).

When in sliding manifold S, the system (7) satisfies the following conditions:

$$s(e) = 0 \tag{9}$$

which is the defining equation for the manifold S and

$$\dot{s}(e) = 0 \tag{10}$$

which is the necessary condition for the state trajectory $e(t)$ of (7) to stay on the sliding manifold S.

Using (7) and (8), the equation (10) can be rewritten as

$$\dot{s}(e) = C\left[Ae + Bv\right] = 0 \tag{11}$$

Solving (11) for v, we obtain the equivalent control law

$$v_{\text{eq}}(t) = -(CB)^{-1} CA\, e(t) \tag{12}$$

where C is chosen such that

$$CB \neq 0.$$

Substituting (12) into the error dynamics (7), we obtain the closed-loop dynamics as

$$\dot{e} = \left[I - B(CB)^{-1} C \right] Ae \tag{13}$$

The row vector C is selected such that the system matrix of the controlled dynamics $\left[I - B(CB)^{-1} C \right] A$ is Hurwitz, *i.e.* it has all eigenvalues with negative real parts. Then the controlled system (13) is globally asymptotically stable.

To design the sliding mode controller for (7), we apply the constant plus proportional rate reaching law

$$\dot{s} = -q\,\mathrm{sgn}(s) - k\,s \tag{14}$$

where $\mathrm{sgn}(\cdot)$ denotes the sign function and the gains $q > 0,\; k > 0$ are determined such that the sliding condition is satisfied and sliding motion will occur.

From equations (11) and (14), we can obtain the control $v(t)$ as

$$v(t) = -(CB)^{-1}\left[C(kI + A)e + q\,\mathrm{sgn}(s)\right] \tag{15}$$

which yields

$$v(t) = \begin{cases} -(CB)^{-1}\left[C(kI + A)e + q\right], & \text{if } s(e) > 0 \\ -(CB)^{-1}\left[C(kI + A)e - q\right], & \text{if } s(e) < 0 \end{cases} \tag{16}$$

Theorem 1. *The master system (1) and the slave system (2) are globally and asymptotically synchronized for all initial conditions* $x(0), y(0) \in R^n$ *by the feedback control law*

$$u(t) = -\eta(x, y) + Bv(t) \tag{17}$$

where $v(t)$ *is defined by (15) and* B *is a column vector such that* (A, B) *is controllable. Also, the sliding mode gains* k, q *are positive.*

Proof. First, we note that substituting (17) and (15) into the error dynamics (4), we obtain the closed-loop error dynamics as

$$\dot{e} = Ae - B(CB)^{-1}\left[C(kI + A)e + q\,\mathrm{sgn}(s)\right] \tag{18}$$

To prove that the error dynamics (18) is globally asymptotically stable, we consider the candidate Lyapunov function defined by the equation

$$V(e) = \frac{1}{2}s^2(e) \tag{19}$$

which is a positive definite function on R^n.

Differentiating V along the trajectories of (18) or the equivalent dynamics (14), we get

$$\dot{V}(e) = s(e)\dot{s}(e) = -ks^2 - q\,\mathrm{sgn}(s)s \tag{20}$$

which is a negative definite function on R^n.

This calculation shows that V is a globally defined, positive definite, Lyapunov function for the error dynamics (18), which has a globally defined, negative definite time derivative $\dot{V}$.

Thus, by Lyapunov stability theory [37], it is immediate that the error dynamics (18) is globally asymptotically stable for all initial conditions $e(0) \in R^n$.

This means that for all initial conditions $e(0) \in R^n$, we have

$$\lim_{t \to \infty} \left\| e(t) \right\| = 0$$

Hence, it follows that the master system (1) and the slave system (2) are globally and asymptotically synchronized for all initial conditions $x(0), y(0) \in R^n$.

This completes the proof.

3. Sliding Controller Design for Global Chaos Synchronization of Identical Hyperchaotic Lorenz Systems

3.1 Theoretical Results

In this section, we apply the sliding mode control results of Section 2 to derive state feedback control laws for the global chaos synchronization of identical hyperchaotic Lorenz systems ([36], Jia, 2007).

Thus, the master system is described by the 4-D Jia dynamics

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_4 \\ \dot{x}_2 &= bx_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -cx_3 + x_1 x_2 \\ \dot{x}_4 &= dx_4 - x_1 x_3 \end{aligned} \tag{21}$$

where x_1, x_2, x_3, x_4 are state variables and a, b, c, d are positive, constant parameters of the system.

The slave system is also described by the controlled 4-D Jia dynamics

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + y_4 + u_1 \\ \dot{y}_2 &= by_1 - y_2 - y_1 y_3 + u_2 \\ \dot{y}_3 &= -cy_3 + y_1 y_2 + u_3 \\ \dot{y}_4 &= dy_4 - y_1 y_3 + u_4 \end{aligned} \tag{22}$$

where y_1, y_2, y_3, y_4 are state variables and u_1, u_2, u_3, u_4 are the controllers to be designed.

The 4-D systems (21) and (22) are hyperchaotic when

$$a = 10, \quad b = 28, \quad c = 8/3 \quad \text{and} \quad d = 1.3$$

Figure 1 illustrates the state orbits of the hyperchaotic Lorenz system.

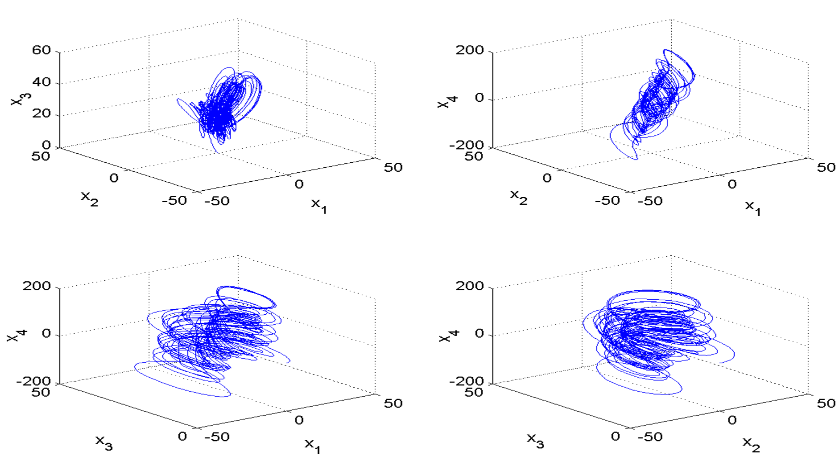

Figure 1. State Orbits of the Hyperchaotic Lorenz System

The chaos synchronization error is defined by

$$e_i = y_i - x_i, \ \ (i = 1, 2, 3, 4) \tag{23}$$

The error dynamics is easily obtained as

$$\begin{aligned}
\dot{e}_1 &= a(e_2 - e_1) + e_4 + u_1 \\
\dot{e}_2 &= be_1 - e_2 - y_1 y_3 + x_1 x_3 + u_2 \\
\dot{e}_3 &= -ce_3 + y_1 y_2 - x_1 x_2 + u_3 \\
\dot{e}_4 &= de_4 - y_1 y_3 + x_1 x_2 + u_4
\end{aligned} \tag{24}$$

We write the error dynamics (24) in the matrix notation as

$$\dot{e} = Ae + \eta(x, y) + u \tag{25}$$

where

$$A = \begin{bmatrix} -a & a & 0 & 1 \\ b & -1 & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}, \ \ \eta(x, y) = \begin{bmatrix} 0 \\ -y_1 y_3 + x_1 x_3 \\ y_1 y_2 - x_1 x_2 \\ -y_1 y_3 + x_1 x_3 \end{bmatrix} \ \text{and} \ \ u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{26}$$

The sliding mode controller design is carried out as detailed in Section 2.

First, we set u as

$$u = -\eta(x, y) + Bv \tag{27}$$

where B is chosen such that (A, B) is controllable.

We take B as

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{28}$$

In the hyperchaotic case, the parameter values are

$$a = 10, \ \ b = 28, \ \ c = 8/3 \ \ \text{and} \ \ d = 1.3$$

The sliding mode variable is selected as

$$s = Ce = \begin{bmatrix} -9 & -3 & -1 & 7 \end{bmatrix} e = -9e_1 - 3e_2 - e_3 + 7e_4 \tag{29}$$

which makes the sliding mode state equation asymptotically stable.

We choose the sliding mode gains as

$$k = 5 \ \ \text{and} \ \ q = 0.2$$

We note that a large value of k can cause chattering and an appropriate value of q is chosen to speed up the time taken to reach the sliding manifold as well as to reduce the system chattering.

From Eq. (15), we can obtain $v(t)$ as

$$v(t) = -6.5e_1 - 17e_2 - 0.3889e_3 + 5.85e_4 - 0.0333\,\text{sgn}(s) \tag{30}$$

Thus, the required sliding mode controller is obtained as

$$u = -\eta(x, y) + Bv \tag{31}$$

where $\eta(x, y), B$ and $v(t)$ are defined as in the equations (26), (28) and (30).

By Theorem 1, we obtain the following result.

Theorem 2. *The identical hyperchaotic Lorenz systems (21) and (22) are globally and asymptotically synchronized for all initial conditions with the sliding mode controller* u *defined by (31).*

3.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic Lorenz systems (21) and (22) with the sliding mode controller u given by (31) using MATLAB.
In the hyperchaotic case, the parameter values are given by

$$a = 10, \quad b = 28, \quad c = 8/3 \quad \text{and} \quad d = 1.3$$

The sliding mode gains are chosen as

$$k = 5 \;\text{ and }\; q = 0.2$$

The initial values of the master system (21) are taken as

$$x_1(0) = 15, \; x_2(0) = 22, \; x_3(0) = 10, \; x_4(0) = 32$$

and the initial values of the slave system (22) are taken as

$$y_1(0) = 10, \; y_2(0) = 6, \; y_3(0) = 31, \; y_4(0) = 28$$

Figure 2 illustrates the complete synchronization of the identical hyperchaotic Lorenz systems (21) and (22).

Figure 3 shows the error states $e_1(t), e_2(t), e_3(t), e_4(t)$ which converge to zero as $t \to \infty$.

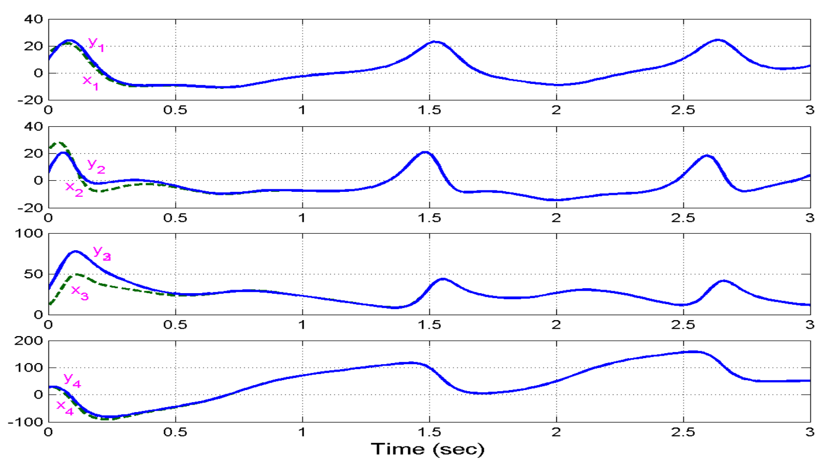

Figure 2. Complete Synchronization of Identical Hyperchaotic Lorenz Systems

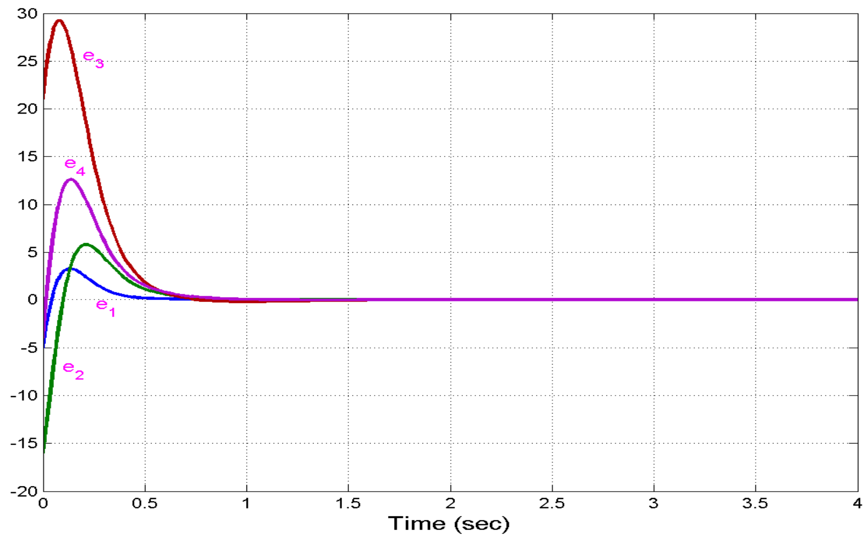

Figure 2. Time Responses of Error States - $e_1(t),\ e_2(t),\ e_3(t),\ e_4(t)$

4. CONCLUSIONS

In this paper, we have derived new results using Lyapunov stability theory for the global chaos synchronization using sliding mode control. As application of our sliding mode controller design, we derived new synchronization schemes for the identical hyperchaotic Lorenz systems (2007). Since the Lyapunov exponents are not required for these calculations, the sliding mode control method is very effective and convenient to achieve global chaos synchronization for the identical hyperchaotic Lorenz systems. Numerical simulations are also shown to illustrate the effectiveness of the synchronization results derived in this paper via sliding mode control.

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