GLOBAL CHAOS SYNCHRONIZATION OF HYPERCHAOTIC PANG AND HYPERCHAOTIC WANG SYSTEMS VIA ADAPTIVE CONTROL

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ABSTRACT

This paper investigates the global chaos synchronization of identical hyperchaotic Wang systems, identical hyperchaotic Pang systems, and non-identical hyperchaotic Wang and hyperchaotic Pang systems via adaptive control method. Hyperchaotic Pang system (Pang and Liu, 2011) and hyperchaotic Wang system (Wang and Liu, 2006) are recently discovered hyperchaotic systems. Adaptive control method is deployed in this paper for the general case when the system parameters are unknown. Sufficient conditions for global chaos synchronization of identical hyperchaotic Pang systems, identical hyperchaotic Wang systems and non-identical hyperchaotic Pang and Wang systems are derived via adaptive control theory and Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is very convenient for the global chaos synchronization of the hyperchaotic systems discussed in this paper. Numerical simulations are presented to validate and demonstrate the effectiveness of the proposed synchronization schemes.

KEYWORDS


1. INTRODUCTION

Chaotic systems are dynamical systems that are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the butterfly effect [1]. Since chaos phenomenon in weather models was first observed by Lorenz in 1963 [2], a large number of chaos phenomena and chaos behaviour have been discovered in physical, social, economical, biological and electrical systems.

A hyperchaotic system is usually characterized as a chaotic system with more than one positive Lyapunov exponent implying that the dynamics expand in more than one direction giving rise to “thicker” and “more complex” chaotic dynamics. The first hyperchaotic system was discovered by Rössler in 1979 [3].
Chaos is an interesting nonlinear phenomenon and has been extensively studied in the last two decades [1-40].

Synchronization of chaotic systems is a phenomenon which may occur when two or more chaotic oscillators are coupled or when a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect which causes the exponential divergence of the trajectories of two identical chaotic systems started with nearly the same initial conditions, synchronizing two chaotic systems is seemingly a very challenging problem.

In 1990, Pecora and Carroll [4] deployed control techniques to synchronize two identical chaotic systems and showed that it was possible for some chaotic systems to be completely synchronized. From then on, chaos synchronization has been widely explored in a variety of fields including physical systems [5], chemical systems [6], ecological systems [7], secure communications [8-10], etc.

In most of the chaos synchronization approaches, the master-slave or drive-response formalism is used. If a particular chaotic system is called the master or drive system and another chaotic system is called the slave or response system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

Since the seminal work by Pecora and Carroll [4], a variety of impressive approaches have been proposed for the synchronization of chaotic systems such as the OGY method [11], active control method [12-16], adaptive control method [17-22], sampled-data feedback synchronization method [23], time-delay feedback method [24], backstepping method [25-26], sliding mode control method [27-32], etc.

In this paper, we investigate the global chaos synchronization of uncertain hyperchaotic systems, viz. identical hyperchaotic Pang systems ([33], 2011), identical hyperchaotic Wang systems ([34], 2006) and non-identical hyperchaotic Pang and hyperchaotic Wang systems. We consider the general case when the parameters of the hyperchaotic systems are unknown.

This paper is organized as follows. In Section 2, we provide a description of the hyperchaotic systems addressed in this paper, viz. hyperchaotic Pang system (2011) and hyperchaotic Wang system (2006). In Section 3, we discuss the adaptive synchronization of identical hyperchaotic Pang systems. In Section 4, we discuss the adaptive synchronization of identical hyperchaotic Wang systems. In Section 5, we discuss the adaptive synchronization of non-identical hyperchaotic Pang and hyperchaotic Wang systems. In Section 6, we summarize the main results obtained in this paper.

2. SYSTEMS DESCRIPTION

The hyperchaotic Pang system ([33], 2011) is described by the dynamics

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= cx_2 - x_1 x_3 + x_4 \\
\dot{x}_3 &= -bx_3 + x_1 x_2 \\
\dot{x}_4 &= -d(x_1 + x_2)
\end{align*}
\]

(1)

where \(x_1, x_2, x_3, x_4\) are the state variables and \(a, b, c, d\) are positive, constant parameters of the system.
The 4-D system (1) is hyperchaotic when the parameter values are taken as

\[ a = 36, \ b = 3, \ c = 20 \quad \text{and} \quad d = 2 \]

The state orbits of the hyperchaotic Pang chaotic system (1) are shown in Figure 1.

Figure 1. State Orbits of the Hyperchaotic Pang Chaotic System

The hyperchaotic Wang system ([34], 2006) is described by

\[
\begin{align*}
\dot{x}_1 &= \alpha(x_2 - x_1) \\
\dot{x}_2 &= \beta x_1 - x_1 x_3 + x_4 \\
\dot{x}_3 &= -\gamma x_3 + \varepsilon x_1^2 \\
\dot{x}_4 &= -\delta x_1
\end{align*}
\]

(2)

where \( x_1, x_2, x_3, x_4 \) are the state variables and \( \alpha, \beta, \gamma, \delta, \varepsilon \) are positive constant parameters of the system.

The 4-D system (2) is hyperchaotic when the parameter values are taken as
The state orbits of the hyperchaotic Wang chaotic system (2) are shown in Figure 2.

![Figure 2. State Orbits of the Hyperchaotic Wang System](image)

### 3. Adaptive Synchronization of Identical Hyperchaotic Pang Systems

#### 3.1 Theoretical Results

In this section, we deploy adaptive control to achieve new results for the global chaos synchronization of identical hyperchaotic Pang systems ([33], 2011), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Pang dynamics described by

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= cx_2 - x_1x_3 + x_4 \\
\dot{x}_3 &= -bx_3 + x_1x_2 \\
\dot{x}_4 &= -d(x_1 + x_2)
\end{align*}
\]

(3)
where \( x_1, x_2, x_3, x_4 \) are the state variables and \( a, b, c, d \) are unknown, real, constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Pang dynamics described by

\[
\begin{align*}
\dot{y}_1 &= a(y_2 - y_1) + u_1 \\
\dot{y}_2 &= cy_2 - y_1y_3 + y_4 + u_2 \\
\dot{y}_3 &= -by_3 + y_1y_2 + u_3 \\
\dot{y}_4 &= -d(y_1 + y_2) + u_4
\end{align*}
\]  

(4)

where \( y_1, y_2, y_3, y_4 \) are the state variables and \( u_1, u_2, u_3, u_4 \) are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

\[
e_i = y_i - x_i, \quad (i = 1, 2, 3, 4)
\]

(5)

The error dynamics is easily obtained as

\[
\begin{align*}
\dot{e}_1 &= a(e_2 - e_1) + u_1 \\
\dot{e}_2 &= ce_2 + e_4 - y_1y_3 + x_1x_3 + u_2 \\
\dot{e}_3 &= -be_3 + y_1y_2 - x_1x_2 + u_3 \\
\dot{e}_4 &= -d(e_1 + e_2) + u_4
\end{align*}
\]

(6)

Let us now define the adaptive control functions

\[
\begin{align*}
u_1(t) &= -\hat{a}(e_2 - e_1) - k_1e_1 \\
v_2(t) &= -\hat{c}e_2 - e_4 + y_1y_3 - x_1x_3 - k_2e_2 \\
v_3(t) &= \hat{b}e_3 + y_1y_2 - x_1x_2 - k_3e_3 \\
v_4(t) &= \hat{d}(e_1 + e_2) - k_4e_4
\end{align*}
\]

(7)

where \( \hat{a}, \hat{b}, \hat{c} \) and \( \hat{d} \) are estimates of \( a, b, c \) and \( d \), respectively, and \( k_i, (i = 1, 2, 3, 4) \) are positive constants.

Substituting (7) into (6), the error dynamics simplifies to

\[
\begin{align*}
\dot{e}_1 &= (a - \hat{a})(e_2 - e_1) - k_1e_1 \\
\dot{e}_2 &= (c - \hat{c})e_2 - k_2e_2 \\
\dot{e}_3 &= -(b - \hat{b})e_3 - k_3e_3 \\
\dot{e}_4 &= -(d - \hat{d})(e_1 + e_2) - k_4e_4
\end{align*}
\]

(8)

Let us now define the parameter estimation errors as
Substituting (9) into (8), we obtain the error dynamics as

\[
\begin{align*}
\dot{e}_1 &= e(a - \hat{a}) - k_1 e_1 \\
\dot{e}_2 &= e_e e_2 - k_2 e_2 \\
\dot{e}_3 &= -e_b e_3 - k_3 e_3 \\
\dot{e}_4 &= -e_d (e_1 + e_2) - k_4 e_4
\end{align*}
\]

(10)

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

\[
V(e_1, e_2, e_3, e_4, e_a, e_b, e_c, e_d) = \frac{1}{2} (e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2),
\]

(11)

which is a positive definite function on \( \mathbb{R}^8 \).

We also note that

\[
\dot{e}_a = -\hat{a}, \quad \dot{e}_b = -\hat{b}, \quad \dot{e}_c = -\hat{c} \quad \text{and} \quad \dot{e}_d = -\hat{d}
\]

(12)

Differentiating (11) along the trajectories of (10) and using (12), we obtain

\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a (e_1 (e_2 - e_1) - \hat{a}) + e_b (-e_2^2 - \hat{b}) \\
+ e_c (e_3^2 - \hat{c}) + e_d (-e_4 (e_1 + e_2) - \hat{d})
\]

(13)

In view of Eq. (13), the estimated parameters are updated by the following law:

\[
\begin{align*}
\dot{a} &= e_1 (e_2 - e_1) + k_5 e_a \\
\dot{b} &= -e_2^2 + k_6 e_b \\
\dot{c} &= e_3^2 + k_7 e_c \\
\dot{d} &= -e_4 (e_1 + e_2) + k_8 e_d
\end{align*}
\]

(14)

where \( k_5, k_6, k_7 \) and \( k_8 \) are positive constants.

Substituting (14) into (13), we obtain

\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_a^2 - k_6 e_b^2 - k_7 e_c^2 - k_8 e_d^2
\]

(15)

which is a negative definite function on \( \mathbb{R}^8 \).
Thus, by Lyapunov stability theory [35], it is immediate that the hybrid synchronization error $e_i, (i = 1, 2, 3, 4)$ and the parameter estimation error $e_o, e_p, e_c, e_d$ decay to zero exponentially with time.

Hence, we have proved the following result.

**Theorem 1.** The identical hyperchaotic Pang systems (3) and (4) with unknown parameters are globally and exponentially synchronized via the adaptive control law (7), where the update law for the parameter estimates is given by (14) and $k_i, (i = 1, 2, \ldots, 8)$ are positive constants. Also, the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ and $\hat{d}(t)$ exponentially converge to the original values of the parameters $a, b, c$ and $d$, respectively, as $t \to \infty$. ■

### 3.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (3) and (4) with the adaptive control law (14) and the parameter update law (14) using MATLAB.

We take

$$k_i = 4 \text{ for } i = 1, 2, \ldots, 8.$$ 

For the hyperchaotic Pang systems (3) and (4), the parameter values are taken as

$$a = 36, \quad b = 3, \quad c = 20, \quad d = 2$$

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0) = 12, \quad \hat{b}(0) = 4, \quad \hat{c}(0) = 2, \quad \hat{d}(0) = 21$$

The initial values of the master system (3) are taken as

$$x_1(0) = 12, \quad x_2(0) = 18, \quad x_3(0) = 35, \quad x_4(0) = 6$$

The initial values of the slave system (4) are taken as

$$y_1(0) = 20, \quad y_2(0) = 5, \quad y_3(0) = 16, \quad y_4(0) = 22$$

Figure 3 depicts the global chaos synchronization of the identical hyperchaotic Pang systems (3) and (4).

Figure 4 shows that the estimated values of the parameters, viz. $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ and $\hat{d}(t)$ converge exponentially to the system parameters

$$a = 36, \quad b = 3, \quad c = 20 \quad \text{and} \quad d = 2$$

as $t \to \infty$. 
Figure 3. Complete Synchronization of Hyperchaotic Pang Systems

Figure 4. Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t)$
4. ADAPTIVE SYNCHRONIZATION OF IDENTICAL HYPERCHAOTIC WANG SYSTEMS

4.1 Theoretical Results

In this section, we deploy adaptive control to achieve new results for the global chaos synchronization of identical hyperchaotic Wang systems ([34], 2006), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Wang dynamics described by

\[
\begin{align*}
\dot{x}_1 &= \alpha(x_2 - x_1) \\
\dot{x}_2 &= \beta x_1 - x_1 x_3 + x_4 \\
\dot{x}_3 &= -\gamma x_3 + \epsilon x_1^2 \\
\dot{x}_4 &= -\delta x_1
\end{align*}
\]

where \(x_1, x_2, x_3, x_4\) are the state variables and \(\alpha, \beta, \gamma, \delta, \epsilon\) are unknown, real, constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Wang dynamics described by

\[
\begin{align*}
\dot{y}_1 &= \alpha(y_2 - y_1) + u_1 \\
\dot{y}_2 &= \beta y_1 - y_1 y_3 + y_4 + u_2 \\
\dot{y}_3 &= -\gamma y_3 + \epsilon y_1^2 + u_3 \\
\dot{y}_4 &= -\delta y_1 + u_4
\end{align*}
\]

where \(y_1, y_2, y_3, y_4\) are the state variables and \(u_1, u_2, u_3, u_4\) are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

\[
\begin{align*}
e_1 &= y_1 - x_1 \\
e_2 &= y_2 - x_2 \\
e_3 &= y_3 - x_3 \\
e_4 &= y_4 - x_4
\end{align*}
\]

The error dynamics is easily obtained as

\[
\begin{align*}
\dot{e}_1 &= \alpha(e_2 - e_1) + u_1 \\
\dot{e}_2 &= \beta e_1 + e_4 - y_1 y_3 + x_1 x_3 + u_2 \\
\dot{e}_3 &= -\gamma e_3 + \epsilon(y_1^2 - x_1^2) + u_3 \\
\dot{e}_4 &= -\delta e_4 + u_4
\end{align*}
\]

Let us now define the adaptive control functions...

\begin{align*}
  u_1(t) &= -\hat{\alpha}(e_2 - e_1) - k_1e_1 \\
  u_2(t) &= -\hat{\beta}e_1 - e_4 + y_1y_3 - x_1x_3 - k_2e_2 \\
  u_3(t) &= \hat{\gamma}e_3 - \hat{\xi}(y_1^2 - x_1^2) - k_3e_3 \\
  u_4(t) &= \hat{\delta}e_1 - k_4e_4 \\
\end{align*}

\begin{equation}
  (20)
\end{equation}

where \(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}\) and \(\hat{\xi}\) are estimates of \(\alpha, \beta, \gamma, \delta\) and \(\xi\), respectively, and \(k_i, (i = 1, 2, 3, 4)\) are positive constants.

Substituting (20) into (19), the error dynamics simplifies to
\begin{align*}
  \dot{e}_1 &= (\alpha - \hat{\alpha})(e_2 - e_1) - k_1e_1 \\
  \dot{e}_2 &= (\beta - \hat{\beta})e_1 - k_2e_2 \\
  \dot{e}_3 &= -(\gamma - \hat{\gamma})e_3 + (\xi - \hat{\xi})(y_1^2 - x_1^2) - k_3e_3 \\
  \dot{e}_4 &= -(\delta - \hat{\delta})e_1 - k_4e_4 \\
\end{align*}

\begin{equation}
  (21)
\end{equation}

Let us now define the parameter estimation errors as
\begin{align*}
  e_{\alpha} &= \alpha - \hat{\alpha}, \quad e_{\beta} = \beta - \hat{\beta}, \quad e_{\gamma} = \gamma - \hat{\gamma}, \quad e_{\delta} = \delta - \hat{\delta} \quad \text{and} \quad e_{\xi} = \xi - \hat{\xi} \\
\end{align*}

\begin{equation}
  (22)
\end{equation}

Substituting (22) into (21), we obtain the error dynamics as
\begin{align*}
  \dot{e}_1 &= e_{\alpha}(e_2 - e_1) - k_1e_1 \\
  \dot{e}_2 &= e_{\beta}e_1 - k_2e_2 \\
  \dot{e}_3 &= -e_{\gamma}e_3 + e_{\xi}(y_1^2 - x_1^2) - k_3e_3 \\
  \dot{e}_4 &= -e_{\delta}e_1 - k_4e_4 \\
\end{align*}

\begin{equation}
  (23)
\end{equation}

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by
\begin{equation}
  V(e_1, e_2, e_3, e_4, e_{\alpha}, e_{\beta}, e_{\gamma}, e_{\delta}, e_{\xi}) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_{\alpha}^2 + e_{\beta}^2 + e_{\gamma}^2 + e_{\delta}^2 + e_{\xi}^2),
\end{equation}

\begin{equation}
  (24)
\end{equation}

which is a positive definite function on \(\mathbb{R}^9\).

We also note that
\begin{align*}
  \dot{e}_{\alpha} &= -\hat{\alpha}, \quad \dot{e}_{\beta} = -\hat{\beta}, \quad \dot{e}_{\gamma} = -\hat{\gamma}, \quad \dot{e}_{\delta} = -\hat{\delta} \quad \text{and} \quad \dot{e}_{\xi} = -\hat{\xi} \\
\end{align*}

\begin{equation}
  (25)
\end{equation}

Differentiating (24) along the trajectories of (23) and using (25), we obtain
In view of Eq. (26), the estimated parameters are updated by the following law:

\[
\begin{align*}
\dot{\alpha} &= e_1(e_2 - e_1) + k_\alpha e_\alpha \\
\dot{\beta} &= e_\beta e_2 + k_\beta e_\beta \\
\dot{\gamma} &= -e_3^2 + k_\gamma e_\gamma \\
\dot{\delta} &= -e_4 e_4 + k_\delta e_\delta \\
\dot{e} &= e_5(y_i^2 - x_i^2) + k_\varepsilon e_\varepsilon
\end{align*}
\]  

(27)

where \( k_i, \ (i = 5, \ldots, 9) \) are positive constants.

Substituting (27) into (26), we obtain

\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_5^2 - k_6 e_6^2 - k_7 e_7^2 - k_8 e_8^2 - k_9 e_9^2
\]

(28)

which is a negative definite function on \( \mathbb{R}^9 \).

Thus, by Lyapunov stability theory [35], it is immediate that the hybrid synchronization error \( e_i, (i = 1, 2, 3, 4) \) and the parameter estimation error \( e_\alpha, e_\beta, e_\gamma, e_\delta, e_\varepsilon \) decay to zero exponentially with time.

Hence, we have proved the following result.

**Theorem 2.** The identical hyperchaotic Wang systems (16) and (17) with unknown parameters are globally and exponentially synchronized via the adaptive control law (20), where the update law for the parameter estimates is given by (27) and \( k_i, (i = 1, 2, \ldots, 9) \) are positive constants.

Also, the parameter estimates \( \dot{\alpha}(t), \dot{\beta}(t), \dot{\gamma}(t), \dot{\delta}(t) \) and \( \dot{e}(t) \) exponentially converge to the original values of the parameters \( \alpha, \beta, \gamma, \delta \) and \( \varepsilon \), respectively, as \( t \to \infty \).

**4.2 Numerical Results**

For the numerical simulations, the fourth-order Runge-Kutta method with time-step \( h = 10^{-6} \) is used to solve the hyperchaotic systems (16) and (17) with the adaptive control law (20) and the parameter update law (27) using MATLAB.

We take

\( k_i = 4 \) for \( i = 1, 2, \ldots, 9 \).

For the hyperchaotic Wang systems (16) and (17), the parameter values are taken as
Suppose that the initial values of the parameter estimates are
\[ \hat{\alpha}(0) = 5, \quad \hat{\beta}(0) = 10, \quad \hat{\gamma}(0) = 7, \quad \hat{\delta}(0) = 14, \quad \hat{\epsilon}(0) = 9 \]

The initial values of the master system (16) are taken as
\[ x_1(0) = 21, \quad x_2(0) = 7, \quad x_3(0) = 16, \quad x_4(0) = 18 \]

The initial values of the slave system (17) are taken as
\[ y_1(0) = 4, \quad y_2(0) = 25, \quad y_3(0) = 30, \quad y_4(0) = 11 \]

Figure 5 depicts the global chaos synchronization of the identical hyperchaotic Wang systems (16) and (17).

Figure 6 shows that the estimated values of the parameters, viz. \( \hat{\alpha}(t) \), \( \hat{\beta}(t) \), \( \hat{\gamma}(t) \), \( \hat{\delta}(t) \) and \( \hat{\epsilon}(t) \) converge exponentially to the system parameters \( \alpha = 10, \beta = 40, \gamma = 2.5, \delta = 10.6 \) and \( \epsilon = 4 \) as \( t \to \infty \).
5. ADAPTIVE SYNCHRONIZATION OF HYPERCHAOTIC PANG AND HYPERCHAOTIC WANG SYSTEMS

5.1 Theoretical Results

In this section, we discuss the global chaos synchronization of non-identical hyperchaotic Pang system ([33], 2011) and hyperchaotic Wang system ([34], 2006), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Pang system described by

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= cx_2 - x_1x_3 + x_4 \\
\dot{x}_3 &= -bx_3 + x_1x_2 \\
\dot{x}_4 &= -d(x_1 + x_2)
\end{align*}
\]

(29)

where \(x_1, x_2, x_3, x_4\) are the state variables and \(a, b, c, d\) are unknown, real, constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Wang dynamics described by

\[
\begin{align*}
\dot{y}_1 &= \alpha(y_2 - y_1) + u_1 \\
\dot{y}_2 &= \beta \dot{y}_1 - y_1y_3 + y_4 + u_2 \\
\dot{y}_3 &= -\gamma y_3 + \epsilon y_1^2 + u_3 \\
\dot{y}_4 &= -\delta y_1 + u_4
\end{align*}
\]

(30)
where \( y_1, y_2, y_3, y_4 \) are the state variables, \( \alpha, \beta, \gamma, \delta, \epsilon \) are unknown, real, constant parameters of the system and \( u_1, u_2, u_3, u_4 \) are the nonlinear controllers to be designed.

The synchronization error is defined by
\[
\begin{align*}
    e_1 &= y_1 - x_1 \\
    e_2 &= y_2 - x_2 \\
    e_3 &= y_3 - x_3 \\
    e_4 &= y_4 - x_4
\end{align*}
\]  
(31)

The error dynamics is easily obtained as
\[
\begin{align*}
    \dot{e}_1 &= \alpha(y_2 - y_1) - a(x_2 - x_1) + u_1 \\
    \dot{e}_2 &= \beta y_1 - cx_2 + e_4 - y_1y_3 + x_1x_3 + u_2 \\
    \dot{e}_3 &= -\gamma y_1 + bx_3 + \epsilon y_1^2 - x_1x_2 + u_3 \\
    \dot{e}_4 &= -\delta y_1 + d(x_1 + x_2) + u_4
\end{align*}
\]  
(32)

Let us now define the adaptive control functions
\[
\begin{align*}
    u_1(t) &= -\hat{\alpha}(y_2 - y_1) + \hat{\alpha}(x_2 - x_1) - k_1 e_1 \\
    u_2(t) &= -\hat{\beta} y_1 + \hat{c}x_2 - e_4 + y_1y_3 - x_1x_3 - k_2 e_2 \\
    u_3(t) &= \hat{\gamma} y_3 - \hat{b}x_3 - \hat{\epsilon} y_1^2 + x_1x_2 - k_3 e_3 \\
    u_4(t) &= \hat{\delta} y_1 - \hat{d}(x_1 + x_2) - k_4 e_4
\end{align*}
\]  
(33)

where \( \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, \hat{\epsilon} \) are estimates of \( \alpha, \beta, \gamma, \delta, \epsilon \) respectively, and \( k_i, (i = 1, 2, 3, 4) \) are positive constants.

Substituting (33) into (32), the error dynamics simplifies to
\[
\begin{align*}
    \dot{e}_1 &= (\alpha - \hat{\alpha})(y_2 - y_1) - (a - \hat{a})(x_2 - x_1) - k_1 e_1 \\
    \dot{e}_2 &= (\beta - \hat{\beta}) y_1 - (c - \hat{c})x_2 - k_2 e_2 \\
    \dot{e}_3 &= -\gamma y_3 + (b - \hat{b})x_3 + (\epsilon - \hat{\epsilon}) y_1^2 - k_3 e_3 \\
    \dot{e}_4 &= -(\delta - \hat{\delta}) y_1 + (d - \hat{d})(x_1 + x_2) - k_4 e_4
\end{align*}
\]  
(34)

Let us now define the parameter estimation errors as
\[
\begin{align*}
    e_a &= a - \hat{a}, \quad e_b = b - \hat{b}, \quad e_c = c - \hat{c}, \quad e_d = d - \hat{d} \\
    e_{\alpha} &= \alpha - \hat{\alpha}, \quad e_{\beta} = \beta - \hat{\beta}, \quad e_{\gamma} = \gamma - \hat{\gamma}, \quad e_{\delta} = \delta - \hat{\delta}, \quad e_{\epsilon} = \epsilon - \hat{\epsilon}
\end{align*}
\]  
(35)

Substituting (35) into (34), we obtain the error dynamics as
\[
\begin{align*}
\dot{e}_1 &= e_d(y_2 - y_1) - e_a(x_2 - x_1) - k_1 e_t \\
\dot{e}_2 &= e_d y_1 - e_c x_2 - k_2 e_2 \\
\dot{e}_3 &= -e_d y_3 + e_y x_2 + e_2 y_1^2 - k_3 e_3 \\
\dot{e}_4 &= -e_d y_1 + e_d (x_1 + x_2) - k_4 e_4 
\end{align*}
\]

We consider the quadratic Lyapunov function defined by

\[
V = \frac{1}{2} \left( e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2 + e_\epsilon^2 \right),
\]

which is a positive definite function on \( R^{13} \).

We also note that

\[
\begin{align*}
\dot{e}_a &= -\dot{a}, \quad \dot{e}_b = -\dot{b}, \quad \dot{e}_c = -\dot{c}, \quad \dot{e}_d = -\dot{d} \\
\dot{\alpha} &= -\dot{\alpha}, \quad \dot{\beta} = -\dot{\beta}, \quad \dot{\gamma} = -\dot{\gamma}, \quad \dot{\delta} = -\dot{\delta}, \quad \dot{\epsilon} = -\dot{\epsilon}
\end{align*}
\]

Differentiating (37) along the trajectories of (36) and using (38), we obtain

\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_a e_a^2 + e_a \left[ -e_t(x_2 - x_1) - \dot{\alpha} \right] + e_y \left[ e_1 y_1 - \dot{\beta} \right] + e_2 \left[ e_2 y_1 - \dot{\gamma} \right] + e_3 \left[ e_3 y_1 - \dot{\delta} \right] + e_4 \left[ e_4 y_1 - \dot{\epsilon} \right]
\]

In view of Eq. (39), the estimated parameters are updated by the following law:

\[
\begin{align*}
\dot{a} &= -e_i(x_2 - x_i) + k_s e_a, \quad \dot{\alpha} = e_i(y_2 - y_i) + k_s e_a \\
\dot{b} &= e_3 x_3 + k_s e_b, \quad \dot{\beta} = e_2 y_1 + k_1 e_\beta \\
\dot{c} &= -e_3 x_2 + k_s e_c, \quad \dot{\gamma} = -e_2 y_3 + k_1 e_\gamma \\
\dot{d} &= e_4 (x_1 + x_2) + k_s e_d, \quad \dot{\delta} = -e_4 y_1 + k_1 e_\delta \\
\dot{e} &= e_3 y_1^2 + k_1 e_\epsilon 
\end{align*}
\]

where \( k_i, (i = 5, \ldots, 13) \) are positive constants.

Substituting (40) into (39), we obtain

\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_a e_a^2 - k_s e_b^2 - k_s e_c^2 - k_s e_d^2 - k_s e_\alpha^2 - k_s e_\beta^2 - k_s e_\gamma^2 - k_s e_\delta^2 - k_s e_\epsilon^2 - k_1 e_\gamma^2 - k_1 e_\delta^2 - k_1 e_\epsilon^2
\]

which is a negative definite function on \( R^{13} \).
Thus, by Lyapunov stability theory [35], it is immediate that the hybrid synchronization error $e_i, (i = 1, 2, 3, 4)$ and all the parameter estimation errors decay to zero exponentially with time.

Hence, we have proved the following result.

**Theorem 3.** The non-identical hyperchaotic Pang system (29) and hyperchaotic Wang system (30) with unknown parameters are globally and exponentially synchronized via the adaptive control law (33), where the update law for the parameter estimates is given by (40) and $k_i, (i = 1, 2, ..., 13)$ are positive constants. Also, the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t), \hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t), \hat{\delta}(t)$ and $\hat{\varepsilon}(t)$ exponentially converge to the original values of the parameters $a, b, c, d, \alpha, \beta, \gamma, \delta$ and $\varepsilon$, respectively, as $t \to \infty$. ■

5.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (29) and (30) with the adaptive control law (33) and the parameter update law (40) using MATLAB. We take $k_i = 4$ for $i = 1, 2, ..., 13$. For the hyperchaotic Pang and hyperchaotic Wang systems, the parameters of the systems are chosen so that the systems are hyperchaotic (see Section 2).

Suppose that the initial values of the parameter estimates are

$$
\begin{align*}
\hat{a}(0) &= 2, \quad \hat{b}(0) = 5, \quad \hat{c}(0) = 10, \quad \hat{d}(0) = 12, \\
\hat{\alpha}(0) &= 7, \quad \hat{\beta}(0) = 9, \quad \hat{\gamma}(0) = 15, \quad \hat{\delta}(0) = 22, \quad \hat{\varepsilon}(0) = 25
\end{align*}
$$

The initial values of the master system (29) are taken as

$$
\begin{align*}
x_1(0) &= 27, \quad x_2(0) = 11, \quad x_3(0) = 28, \quad x_4(0) = 6
\end{align*}
$$

The initial values of the slave system (30) are taken as

$$
\begin{align*}
y_1(0) &= 10, \quad y_2(0) = 26, \quad y_3(0) = 9, \quad y_4(0) = 30
\end{align*}
$$

Figure 7 depicts the global chaos synchronization of hyperchaotic Pang and hyperchaotic Wang systems. Figure 8 shows that the estimated values of the parameters, viz. $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t), \hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t), \hat{\delta}(t)$ and $\hat{\varepsilon}(t)$ converge exponentially to the system parameters $a = 36, b = 3, c = 20, d = 2, \alpha = 10, \beta = 40, \gamma = 2.5, \delta = 10.6$ and $\varepsilon = 4$, respectively, as $t \to \infty$. 

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Figure 7. Complete Synchronization of Hyperchaotic Pang and Wang Systems

Figure 8. Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t), \hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t)$
6. CONCLUSIONS

In this paper, we have derived new results for the adaptive synchronization of identical hyperchaotic Pang systems (2011), identical hyperchaotic Wang systems (2006) and non-identical hyperchaotic Pang and hyperchaotic Wang systems with unknown parameters. The adaptive synchronization results derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is a very effective and convenient for achieving global chaos synchronization for the uncertain hyperchaotic systems discussed in this paper. Numerical simulations are given to illustrate the effectiveness of the adaptive synchronization schemes derived in this paper for the global chaos synchronization of identical and non-identical uncertain hyperchaotic Pang and hyperchaotic Wang systems.

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