Radio Number Of Wheel Like Graphs

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Abstract.

In this paper we establish the radio number for Flower Wheel graph (F W^k_n), k-Wheel graph (kW ) and Joint-Wheel graph(W H_n).

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1. INTRODUCTION

A radio labeling is an assignment of labels, traditionally represented by integers, to the vertices of a graph. Formally, for a given graph G = (V, E) with V being the set of vertices and E being the set of edges, a radio labeling is a function from the vertices of the graph to some subset of positive integers.

For a set of given stations, the task is to assign to each city a channel, which is a non-negative integer, so that interference is prohibited and the span of the channel assigned is minimized. Hale was the first who proposed graph to model these channel assignment in 1980 [5]. Later in 2001 Chartrand, Erwin, Zhang, and Harary were motivated by regulations for channel assignments of FM radio stations to introduce the radio labeling of graphs [1]. Usually, the level of interference between any two stations is closely related to the geographic locations of the station, the closer are the stations the stronger is the interference. Suppose we consider two levels of interference, major and minor. Major interference occurs between two very close stations; to avoid it, the channel assigned to a pair of very close

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stations have to be at least two apart. Manhat interference occurs between close stations; to avoid it, the channel assigned to a pair of close stations should be different. To model this problem, we construct a graph $G$ by representing each station by a vertex and connecting two vertices by an edge if the geographical locations of the corresponding stations are very close. Two close stations are represented by, in the corresponding graph $G$, a pair of vertices that are distance two apart.

For a simple graph $G$, let $\text{diam}(G)$ denote the diameter of $G$ which is the maximum shortest distance between two distinct vertices. For any two vertices $u$ and $v$ in $G$, let $d(u, v)$ denote the smallest distance between $u$ and $v$. Radio labeling (multi-level distance labeling or distance labeling) for $G$ is a one-to-one mapping $f: V(G) \rightarrow \mathbb{Z}$ satisfying the condition

$$d(u, v) + |f(u) - f(v)| \geq 1 + \text{diam}(G) \quad (1.1)$$

for all $u, v \in V(G)$. The span of a labeling $f$ is the maximum integer that maps to a vertex of a graph $G$. The radio number ($rn(G)$) of $G$ is the lowest span over all radio labelings of the graph. We will refer to inequality (1.1) as the radio condition. Note that this condition necessitates the use of distance integers, thus $rn(G) \geq |V(G)|$ for all graphs $G$. Radio labeling are sometimes referred to as multi-distance labeling and they are equivalent to $k$-labeling for $k = \text{diam}(G)$. In this paper we will consider simple and undirected graph.

2. Some Known Results

In this section we recall some known results about the radio number of graphs. Chartrand, Erwin, and Zhang [1] gave the upper bound for the radio number of Path($P_n$).

**Theorem 2.1.** [1] For any positive integer $n$,

$$rn(P_n) \leq \begin{cases} 2k^2 + k, & \text{if } n = 2k + 1; \\ 2(k^2 - k) + 1, & \text{if } n = 2k; \end{cases}$$

where $P_n$ is the Path on $n$ vertices. Moreover, the bound is sharp when $2 \leq n \leq 5$.

The exact value for the radio number of Path was given by Liu, and Zhu [8].

**Theorem 2.2.** [8] For any $n \geq 4$,

$$rn(P_n) = \begin{cases} 2k^2 + 2, & \text{if } n = 2k + 1; \\ 2k(k - 1) + 1, & \text{if } n = 2k. \end{cases}$$

Also, Liu and Zhu [8] gave the radio number for Cycle($C_n$).

**Theorem 2.3.** [8] Let $C_n$ be an $n$-vertex Cycle. For $n \geq 3$ we have
However Chartrand, Erwin, Harary, and Zhang [2] obtained different values than Liu and Zhu [8]. They found the lower and upper bound for the radio number of Cycle($C_n$).

**Theorem 2.4.** [2] For $k \geq 3$,

\[
\begin{cases}
\frac{n-2}{2} \phi(n) + 1, & \text{if } n \equiv 0, 2 \pmod{4}; \\
\frac{n-1}{2} \phi(n), & \text{if } n \equiv 1, 3 \pmod{4},
\end{cases}
\]

where

\[
\phi(n) = \begin{cases} 
k + 1, & \text{if } n = 4k + 1; \\
k + 2, & \text{if } n = 4k + r \text{ for } r = 0, 2, 3.
\end{cases}
\]

Liu [7] gave the lower bound for the radio number of Tree($T_n$).

**Theorem 2.5.** [7] If $T_n$ is an $n$-vertex rooted tree with diameter $d$. Then

\[
\text{rn}(T_n) \geq (n-1)(d+1) + 1 - 2w(T_n),
\]

where $w(T_n)$ represents the weight.

The exact value for the radio number of Hypercube($Q_n$) was given by R. Khennoufa and O. Togni [6].

**Theorem 2.6.** [6] For any positive integer $n \geq 1$,

\[
\left\lfloor \frac{n}{2} \right\rfloor - 1, \quad \text{for } n \geq 6.
\]

M.M. Rivera, M. Tomova, C. Wyels, and A. Yeager [10] gave the radio number of $C_n \square C_n$, where denotes the Cartesian product.

**Theorem 2.7.** [10] For any non-negative integer $k$, we have

\[
\text{rn}(C_{2k} \square C_{2k}) = 2k^3 + 4k^2 - k
\]

and

\[
\text{rn}(C_{2k+1} \square C_{2k+1}) = 2k^3 + 4k^2 + 2k - 1.
\]
In [3] C. Fernandez, A. Flores, M. Tomova, and C. Wyels worked on finding the radio number for Complete graph, Stargraph, Complete Bipartite graph, Wheelgraph and Geargraph. They have proved the following results:

- $r_n(K_n) = n$.
- $r_n(S_n) = n + 2$.
- $r_n(K_{m,n}) = m + n + 1$.
- $r_n(W_n) = n + 2$ for $n \geq 5$.
- $r_n(G_n) = 4n + 2$ for $n \geq 4$.

M. T. Rahimand I. Tomescu [9] investigated the radio number of Helm graph $(H_n)$. They proved the following result.

**Theorem 2.8.** [9] Let $H_n$ be a Helm graph. For $n \geq 5$ we have

$$r_n(H_n) = 4n + 2,$$

where $n$ denotes the number of vertices in a cycle.

### 3. New Result

The radio number of Flower Wheel graph $(FW^k)$: In this section we will find the radio number by examining labels which have minimum distance between them. For an upperbound, we find a specific radiolabeling which gives us span equal to the lowerbound. FlowerWheelgraph consist of $k$ disjoint copies of Wheel graph $(W_n)$ meeting in a common vertex (different from hub). The common vertex of all the copies of Wheel is named as the central vertex. Its clear that $FW^k$ has $(t+3)k+1$ vertices and $diam(FW^k) = 4$ for all $n \geq 5$, where $n$ is the number of vertices in one copy of Wheel graph. We denote the number of vertices (in one copy of Wheel) which are non-adjacent to the central vertex by $t$. We consider the case when all the copies of Wheel graph have same number of vertices.

The labeling of $FW^k$ is defined as follows:

To establish the radio number of $FW^k$ we will refer to a labeling of the vertices $\{z, v_1, v_2, \ldots, v_{2k}, v_{2k+1}, v_{2k+2}, \ldots, v_{3k}, u_1, u_2, \ldots, u_{tk}\}$ of $FW^k$ that distinguishes the vertices by their characteristics. The central vertex is labeled as $z$, the vertices adjacent to the central are labeled sequentially by $\{v_1, v_2, \ldots, v_{2k}, v_{2k+1}, v_{2k+2}, \ldots, v_{3k}\}$ in a clockwise direction. From Figure 2 it is clear that firstly we label $\{v_1, v_2, \ldots, v_{2k}\}$ where $v_1$ is not the hub vertex, and after labeling these vertices we label $\{v_{2k+1}, v_{2k+2}, \ldots, v_{3k}\}$ (which are actually the hub vertices). Vertices which are not adjacent to $z$ are labeled sequentially by $\{u_1, u_2, \ldots, u_{tk}\}$ in a clockwise direction. We specify $u_1$ adjacent to $v_1$ and $v_{2k+1}$.
First of all we will find the radio number of $FW_k$. It's a special case of $FW_k$ when $t = 0$, where $t$ is number of vertices (in one copy) which are non-adjacent to $z$. We can follow the above procedure to label the vertices of $FW_k$.

**Theorem 3.1.** For $k \geq 2$, $r_n(FW_k) = 3k + 2$. 

**Proof.** First of all we will find the lower bound for the radio number of $FW_k$.

Lowerbound for $r_n(FW_k)$: Assume $k \geq 2$. Since $diam(FW_k) = 2$, any radio labeling $f$ of $FW_k$ must satisfy the radio condition i.e.

$$d(u, v) + |f(u) - f(v)| \geq 1 + diam(FW_k) \geq 3$$

hold for all distinct $u, v \in V(FW_k)$. To determine the lower bound we have to count the minimum number of restricted values associated with the vertices of $FW_k$. Let $f(z) = a$, where $a \in \mathbb{Z}^+$. Since $d(z, v_i) = 1$, where $z = v_i$ for $1 \leq i \leq 3k$. The radio condition becomes

$$d(z, v_i) + |f(z) - f(v_i)| \geq 1 + diam(FW_k), or 1 + |f(z) - f(v_i)| \geq 3, or |f(z) - f(v_i)| \geq 2.$$ 

So, there exists one restricted value associated with $z$. If $d(v_i, v_j) \leq 2$, where $1 \leq i, j \leq 3k$, then the radio condition becomes

$$d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 1 + diam(FW_k), or 2 + |f(v_i) - f(v_j)| \geq 3, or |f(v_i) - f(v_j)| \geq 1.$$ 

So, we can assign the consecutive integers to the following sets $\{v_1, v_3, \ldots, v_{2k-1}\}, \{v_2, v_4, \ldots, v_{2k}\}$ and $\{v_{2k+1}, v_{2k+2}, \ldots, v_{3k}\}$ respectively. Therefore, there exist no restricted value associated with $v_i$ for $1 \leq i \leq 3k$. Hence, there is only one restricted value associated with any label of $FW_k$. Thus, $r_n(FW_k) \geq allowed values + restricted value$

**Upper bound for** $r_n(FW_k)$: If $f$ is any radio labeling of $FW_k$, then span of this labeling will provide an upper bound for the radio number of $FW_k$. In order to find an upper bound we define a radio labeling $f : V(FW_k) \rightarrow \mathbb{Z}^+$ as follows:
\[ f(z) = 1, \]
\[ f(v_{2i-1}) = 2 + i, \quad \text{for} \quad 1 \leq i \leq k, \]
\[ f(v_{2i}) = 2 + k + i, \quad \text{for} \quad 1 \leq i \leq k, \]
\[ f(v_{2k+i}) = 2(1 + k) + i, \quad \text{for} \quad 1 \leq i \leq k. \]

**Figure 1.** Radio labeling of FW^k_4

**Claim:** The labeling \( f \) is a valid radio labeling. We have to show that the radio condition

\[ d(u, v) + |f(u) - f(v)| \geq 1 + \text{diam}(FW^k_4) \geq 3 \]

holds for all distinct \( u, v \in V(FW^k_4) \). We will discuss two cases:

**Case 1:** Since \( d(z, v_i) = 1 \), where \( 1 \leq i \leq 3k \) and \( f(z) = 1, f(v_i) \geq 3 \). The radio condition in this case will be

\[ d(z, v_i) + |f(z) - f(v_i)| \geq 1 + |1 - 3|, \quad \text{or} \quad d(z, v_i) + |f(z) - f(v_i)| \geq 3. \]

Hence, the radio condition is satisfied.

**Case 2:** Since \( d(v_i, v_j) \leq 2 \), where \( l \leq i, j \leq 3k \) and \( f(v_i) \geq 3 \). The possible label difference for each pair will satisfy \( |f(v_i) - f(v_j)| \geq 1 \). The radio condition in this case will be

\[ d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 2 + 1, \quad \text{or} \quad d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 3. \]

Hence, the radio condition is satisfied.

These two cases establish the claim that \( f \) is a valid radio labeling of \( FW^k_4 \).

Thus, \( \text{rn}(FW^k_4) \leq \text{span}(f) = 3k + 2 \).

From the lower and upper bound of \( \text{rn}(FW^k_4) \), we have \( \text{rn}(FW^k_4) = 3k + 2 \).
An example of radio labeling of FW$^4$ is shown in Figure 1. In the next theorem we will find the lower bound for the radio number of FW$^k$.

**Theorem 3.2.** For $k \geq 4$ and $n \geq 5$,

$$rn(FW^k_n) \geq tk + 9k + 2,$$

where $k$ is the number of copies of Wheel, $n$ is the number of vertices in each copy of the Wheel and $t$ be the number of vertices which are non-adjacent to the central vertex.

**Proof.** Assume $k \geq 4$. Since diam($FW^k_n$) = 4, so any radio labeling $f$ of $FW^k_n$ must satisfy the radio condition i.e.

$$d(u, v) + |f(u) - f(v)| \geq 5$$

holds for all distinct $u, v \in V(FW^k_n)$. Now we count the total number of restricted values:

**Restricted values associated with any label of $z$:** If $z$ is label as $a$, then as $d(z, u_i) = 2$ for $1 \leq i \leq tk$, where $z = u_i$ for all $u_i$ non-adjacent with $z$, the radio condition becomes

$$d(z, u_i) + |f(z) - f(u_i)| \geq 1 + \text{diam}(FW^k_n),$$

or $2 + |f(z) - f(u_i)| \geq 1 + 4$, or $|f(z) - f(u_i)| \geq 3$. Hence, the number of restricted values associated with any label of $z$ are 2.

**Restricted value associated with any label of the vertices non-adjacent to $z$:** Since $d(u_i, u_j) \leq 4$, when $i = j$ and for all $1 \leq i, j \leq tk-1$. The radio condition becomes

$$d(u_i, u_j) + |f(u_i) - f(u_j)| \geq 5, or 4 + |f(u_i) - f(u_j)| \geq 5, or |f(u_i) - f(u_j)| \geq 1.$$ It means we can assign consecutive integers to $u_i$, which implies that there are no restricted value associated with any label of $u_i$.

**Restricted value associated with any label of the vertex $u_{tk}$ non-adjacent to $z$:** Suppose $d(u_{tk}, v_i) \leq 3$ for $1 \leq i \leq 3k$, where $u_{tk} = v_i$ and for all $v_i$ adjacent to $z$, the radio condition in this case will be

$$d(u_{tk}, v_i) + |f(u_{tk}) - f(v_i)| \geq 5, or 3 + |f(u_{tk}) - f(v_i)| \geq 5, or |f(u_{tk}) - f(v_i)| \geq 2.$$ So, there is only one restricted value corresponding to $u_{tk}$.

**Restricted values associated with any label of the vertices adjacent to the central vertex:** Since $v_i$ denote any vertex adjacent to $z$. If $d(v_i, v_j) \leq 2$, when $v_i = v_j$ for $1 \leq i, j \leq 3k$. Then, the radio condition becomes

$$d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 1 + 4, or 2 + |f(v_i) - f(v_j)| \geq 5, or |f(v_i) - f(v_j)| \geq 3.$$ Therefore, restricted values associated with each label of $v_i$ are 2. Since we have two restricted values for each $3k-1$ vertices. Hence, the total restricted values in this case will be $2(3k - 1)$. 


Total number of restricted values associated with any label of $FW_k$;

Total number of restricted values associated with any label of $FW_k$ will be the sum of restricted value associated with $z+$ restricted value associated with $u_{i+}$ restricted value associated with $u_{tk+}$ restricted value associated with $v_i = 2 + 0 + 1 + 2(3k - 1) = 6k + 1$.

Hence, $rn(FW_k) \geq \text{allowed values + restricted values}$

$$\geq (t + 3)k + 1 + 6k + 1$$

$$= tk + 9k + 2.$$ 

Hence, we establish the lower bound for the radio number of $FW_k$.

Figure 2. Relabeling and Radio labeling of $FW_4$

Our next result will give the upper bound for the radio number of $FW_k$.

**Theorem 3.3.** For $k \geq 4$ and $n \geq 5$, $rn(FW_k) \leq tk + 9k + 2$.

**Proof.** If $f$ is any radio labeling of $FW_k$, then span of this labeling will provide an upper bound for the radio number of $FW_k$. In order to find an upper bound firstly we define the position function $p$ that renames the
vertices of $FW^k_n$ using the set $\{x_0, x_1, \ldots, x_{(t+3)k}\}$. Then we specify the
labels $f(x_i)$ so that $i < j$ if and only if $f(i) < f(j)$. For $k \geq 2$ and $n \geq 5$,
the position function $p : V(FW^k_n) \rightarrow \{x_0, x_1, \ldots, x_{(t+3)k}\}$ is defined as
follows:

\[ p(z) = x_0. \]

For $1 \leq j \leq t$, $p(u_{f(i+j-1)}) = x_{i+(j-1)k}$, where $1 \leq i \leq k$
and $p(v_i) = x_{tk+i}$ for $1 \leq i \leq (t+3)k$.

Next, we define a radio labeling $f : \{x_0, x_1, \ldots, x_{(t+3)k}\} \rightarrow Z^+$ as follows:

\[
f(x_i) = \begin{cases} 
1, & \text{for } i = 0; \\
3 + i, & \text{for } 1 \leq i \leq tk; \\
tk + 2 + 3(i - tk), & \text{for } tk + 1 \leq i \leq (t + 3)k.
\end{cases}
\]

Claim: The labeling $f$ is a valid radio labeling. We have to show that the radio condition

\[ d(u, v) + |f(u) - f(v)| \geq 1 + \text{diam}(FW^k_n) \geq 5 \]

must hold for all pair of vertices $(u, v)$, where $u = v$.

Case 1: Consider the pair $(z, r)$, when $z = r$ for all $r \in V(FW^k_n)$. Since $d(z, r) \leq 2$, $p(z) = x_0$ and $p(r) = x_i$ for $1 \leq i \leq (t + 3)k$. Therefore, $f(x_i) \geq 4$ for all $1 \leq i \leq (t + 3)k$ and $f(z) = 1$. So, the radio condition becomes

\[ d(z, r) + |f(z) - f(r)| \geq 2 + 1 = 3. \]

Hence, the radio condition is satisfied.

Case 2: Consider the pair of vertices $(v_i, v_j)$, where $1 \leq i, j \leq 3k$. As $d(v_i, v_j) \leq 2$, the label difference for each pair will be

\[ |f(v_i) - f(v_j)| = |f(x_{tk+i}) - f(x_{tk+j})| = |tk + 2 + 3(i - tk) - tk - 2 - 3(j - tk)|. \]

$|f(v_i) - f(v_j)| = 3|i - j| \geq 3$. The radio condition becomes

\[ d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 2 + 3 = 5. \]

Hence, the radio condition is satisfied.

Case 3: Since $d(u_i, u_w) \leq 4$ for $1 \leq i, w \leq tk$, therefore

\[ |f(u_i + t(i - 1)) - f(u_j + t(w - 1))| = |f(x_i + (j - 1)k) - f(x_w + (j - 1)k)| = |i + (j - 1)k - w - (j - 1)k|. \]

Hence, the radio condition becomes

\[ d(u_i, u_w) + |f(u_i) - f(u_w)| \geq 4 + 1 = 5. \]

Hence, the radio condition is satisfied.
**Case 4:** Consider the pair \((v_i, u_w)\), where \(i = w\). As \(d(v_i, u_w) \leq 3\) for \(1 \leq i \leq 3k\) and \(1 \leq w \leq tk\). We have \(f(u_w) \in \{4, 5, \ldots, tk + 3\}\) and \(f(v_i) \in \{tk + 5, tk + 8, \ldots, tk + 9k + 2\}\). The possible label difference for each pairs are,

\[|f(v_i) - f(u_w)| = |tk + 5 - tk - 3| = 2,\]
\[|f(v_i) - f(u_w)| = |tk + 9k + 2 - 4| = tk + 9k - 2.\]

So, \(|f(v_i) - f(u_w)| \geq 2\). The radio condition becomes

\[d(v_i, u_w) + |f(v_i) - f(u_w)| \geq 3 + 2 = 5.\]

Hence, the radio condition is satisfied. These four cases establish the claim that \(f\) is a valid radio labeling of \(FW_k\).

Thus, \(rn(FW_k) \leq \text{span}(f) = tk + 9k + 2.\)

An example of radio labeling of \(FW_4\) is shown in Figure 2.
**Proof.** Since $\text{diam}(kW) = 2$ for any positive integer $k$. We have the radio condition $d(u, v) + |f(u) - f(v)| \geq 3$ for all distinct $u, v \in V(kW)$, where $f$ is the radio labeling of $kW$. First of all we will count the minimum number of restricted labels which will eventually give us the lower bound for the radio number of $kW$.

**Restricted value associated with** $z$: Let us take $f(z) = b$, where $b \in \mathbb{Z}^+$. Since $d(z, v_j) = 1$ for all $z = v_j$, where $1 \leq j \leq l_i$, then $b + 1$ is the restricted label associated with $z$.

**Restricted value associated with** $v_j$: As $d(v_j, v_w) \leq 2$ for all $v_j = v_w$, where $1 \leq j, w \leq l_i$, we will have two cases.

When $l_1 = l_2 = \cdots = l_k$: There exist no restricted value associated with $v_j$, where $1 \leq j \leq l_i$ i.e. we can assign the consecutive integer to $v_j$.

When $l_1 \geq l_2 \geq \cdots \geq l_k$: There exist no restricted value associated with $v_j$, where $1 \leq j \leq l_i$ i.e. we can assign the consecutive integer to $v_j$.

So, there exist only one restricted value associated with any label of $kW$.

The total number of allowed labels are,

$l_i + 1$ for $l_1 = l_2 = \cdots = l_k$ and $l_1 \geq l_2 \geq \cdots \geq l_k$.

Hence, the radio number of $kW \geq$ allowed values + restricted value

i.e. $\text{rn}(kW) \geq$

which establish the lower bound for $\text{rn}(kW)$.

Our next theorem will give the upper bound for $\text{rn}(kW)$.

**Theorem 3.6.** For $k \geq 2$, we have

where $l_i$ are the length of the concentric cycles.

**Proof.** We will define our radio labeling $f : V(kW) \to \mathbb{Z}^+$ which will have a minimum span = $k \cdot l_i + 2$ is defined as follows:

Step 1: We start labeling from the central vertex $z$. Let $f(z) = 1$.

Step 2: After labeling $z$ we move to $v_j$, where $1 \leq j \leq l_i$. We can start labeling from any $v_j$ let $f(v_j) = 3$. In order to label $v_j$ we will consider those vertices which has distance two between them i.e. if $d(v_j, v_w) = 2$, where $1 \leq j, w \leq l_i$ and $j = w$ then we can assign consecutive integer to $v_j$ and $v_w$ so that the radio condition is satisfied. For $k = 3$ the radio labeling of $kW$ defined above is illustrated in Figure 3.
Claim: $f$ is a radio labeling. We must show that the radio condition $d(u, v) + |f(u) - f(v)| \geq \text{diam}(G) + 1 \geq 3$ holds for all pair of vertices $(u, v)$ (where $u = v$). We will have two cases:

**Case 1:** Consider the pair $(z, v_j)$, where $z = v_j$ for $1 \leq j \leq l_i$. Since $d(z, v_j) = 1$, $f(z) = 1$ and $f(v_j) \geq 3$ for all $1 \leq j \leq l_i$. Examining the label difference for each pair, we have $|f(z) - f(v_j)| \geq 2$. So, the radio condition becomes $d(z, v_j) + |f(z) - f(v_j)| \geq 1 + 2 = 3$. Hence, the radio condition is satisfied in this case.

**Case 2:** Consider the pair $(v_j, v_w)$, where $j = w$ and $1 \leq j, w \leq l_i$. Since $f(v_j) \geq 3$ and $d(v_j, v_w) \leq 2$. So, the label difference will be $|f(v_j) - f(v_w)| \geq 1$ for all distinct $v_j, v_w$. The radio condition for such pair of vertices becomes $d(v_j, v_w) + |f(v_j) - f(v_w)| \geq 2 + 1 = 3$. Hence, the radio condition is satisfied.

These two cases establish the claim that $f$ is a valid radio labeling of $kW$. Thus, $\text{rn}(kW) \leq \text{span}(f) = l_1 + 2$.

**Note:** In Figure 3 when $l_1 = l_2 = l_3 = 4$ we start labeling from $v_6$ i.e. $f(v_6) = 3$. After labeling $v_6$ we move to $v_1$ because $d(v_6, v_1) = 2$ so, we can assign the consecutive integer to $v_1$ i.e. $f(v_1) = 4$. After labeling $v_1$ we move to $v_8$ because $d(v_8, v_1) = 2$ so, we can assign the consecutive integer to $v_8$ i.e. $f(v_8) = 5$. We continue in the same way and label all the vertices of $kW$. Similarly, when $l_1 = 6, l_2 = 4$ and $l_3 = 3$ we start labeling from $v_{10}$ i.e. $f(v_{10}) = 3$. After labeling $v_{10}$ we move to $v_6$ because $d(v_{10}, v_6) = 2$ so, we can assign the consecutive integer to $v_6$ i.e. $f(v_6) = 4$. After labeling $v_6$ we move to $v_{11}$ because $d(v_{11}, v_6) = 2$ so, we can assign the consecutive integer to $v_{11}$ i.e. $f(v_{11}) = 5$. We continue in the same way and label all the vertices of $kW$. 
**Theorem 3.7.** If $kW$ is a $k$-Wheel graph, then

$$rn(kW) = \sum_{i=1}^{k} l_i + 2,$$

where $l_i$ are the length of the concentric cycles.

**Proof.** Follows from Theorem 3.5 and Theorem 3.6.

The radio number of Joint-Wheel graph ($WH_n$): Joint-Wheel graph ($WH_n$) is defined as follows: It consists of two disjoint copies of Wheel which are joined by an edge between two rim vertices. It is easy to note that $WH_n$ has $2n + 2$ vertices and $4n + 1$ edges, where $n$ is the number of rim vertices in one copy of the Wheel graph. It is easy to see that for $n \geq 4$, $\text{diam}(WH_n) = 5$.

The labeling of Joint-Wheel is defined as follows:

To establish the radio number of Joint-Wheel we will define a labeling for the vertices of $WH_n$ that distinguishes the vertices by their characteristics. The hub vertices are labeled as $z_1$ and $z_2$, the vertices adjacent to $z_1$ and $z_2$ are labeled sequentially by $\{v_1, v_2, \ldots, v_n\}$ in counterclockwise direction and by $\{u_1, u_2, \ldots, u_n\}$ in clockwise direction respectively. We specify that $v_1, v_n - 1$ are adjacent to $v_n$ and $u_1, u_n - 1$ are adjacent to $u_n$, also $v_n$ and $u_n$ are the end vertices of the bridge (between two copies of Wheel graph).

**Theorem 3.8.** For every $n \geq 10$, $rn(WH_n) \geq 4n + 7$.

**Proof.** Since $\text{diam}(WH_n) = 5$, we must show that the radio condition $d(u, v) + |f(u) - f(v)| \geq 6$ holds for every two distinct vertices $u, v \in V(WH_n)$. We start labeling from the vertices $v_2$ and $u_2$. If we assume that $f(v_2) = a$ and $f(u_2) = a + 1$. Then it may be noted that whenever we assign an integer to one copy of Wheel, we must assign the next possible integer to the second copy of Wheel. We will discuss even and odd cases separately.

**When n is even:** Since we start labeling from $v_2$ and $u_2$, i.e., $f(v_2) = a$ and $f(u_2) = a + 1$. So, there exist no restricted value associated with $v_2$. After labeling $u_2$, we move to $v_i$, where $1 \leq i \leq n$ and $i = 2$, if $f(v_n) = a + b$ then there exist two restricted values associated with $u_2$. After assigning label to $v_n$ we label $u_3$, i.e., $f(u_3) = a + 7$. So, there are two restricted values associated with $v_n$. Following in the similar way we can see that there exist two restricted values associated with each vertex of the following set $\{u_2, v_n, v_3, u_n, u_4\}$.

Consider the pair $(v_{2i+2}, u_{2j-1})$. Since $d(v_{2i+2}, u_{2j-1}) \leq 5$, where $2 \leq i \leq l \leq j \leq 2$ and $j = 2$. If $f(v_{2i+2}) = b$ (the value of $b$ must be greater than previously assign integer) then $f(u_{2j-1}) = b + 2$ i.e., $b + 1$ is the restricted value for the remaining $v_i$ and $u_j$, where $3 \leq i \leq l$ and $3 \leq j \leq n$.
So, there exist one restricted value associated with v6. Now we will move to the first copy of wheel. After labeling \( f(u_1) = b + 2 \) we assign \( b + 4 \) to v8 i.e. there exist one restricted value associated with \( u_1 \). Following in the similar way we can see that there exist one restricted value associated with each \( v_{2i+2} \) and \( u_{2j} - 1 \), where \( 2 \leq i, j \leq 2 \). Therefore, total number of restricted values associated with \( v_{2i+2} \) are and restricted values associated with \( u_{2j} - 1 \) are. Similarly we consider the pair of vertices restricted values associated with \( u_{2j+2} \) are

Since \( d(u_n-2, z_1) = 4 \), the radio condition becomes

\[
d(u_n-2, z_1) + |f(u_n-2) - f(z_1)| \geq 6, \text{ or } |f(u_n-2) - f(z_1)| \geq 2.
\]

which implies that there exist two restricted values associated with \( u_{n-2} \). If \( z_1 \) is labeled as \( c \) (the value of \( c \) must be greater than previously assigned integer), then any positive value from the set \{ \( c+1, c+2 \) \} assigned to \( z_2 \) will not satisfy the radio condition which is defined above for the pair of vertices \( (z_1, z_2) \). So, there are two restricted values associated with \( z_1 \).

Therefore, the total number of restricted values will be the sum of restricted values associated with \{ \( u_2, v_n, v_3, u_n, u_4 \) \} + restricted values associated with \( v_{2i+2} \) + restricted values associated with \( u_{2j-1} \) + restricted values associated with \( v_{2i-1} \) + restricted values associated with \( u_{2j+2} \) + restricted values associated with \( u_{n-2} \) + restricted values associated with \( z_1 \)

When \( n \) is odd: Since we start labeling from \( v_2 \) and \( u_2 \) i.e. \( f(v_2) = a \) and \( f(u_2) = a + 1 \). So, there exist no restricted value associated with \( v_2 \). After labeling \( u_2 \) we move to \( v_i \), where \( 1 \leq i \leq n \) and \( i = 2 \), if \( f(v_i) = a + 4 \) then there exist two restricted values associated with \( u_2 \). After assigning label to \( v_n \) we label \( u_3 \) i.e. \( f(u_3) = a + 7 \). So, there are two restricted values associated with \( v_n \). Following in the similar way we can see that there exist two restricted values associated with each vertex of the following set \{ \( u_2, v_n, v_3, u_n, u_4 \) \}.

Consider the pair \( (v_{2i+2}, u_{2j} - 1) \). Since \( d(v_{2i+2}, u_{2j} - 1) \leq \frac{n-1}{2} \) where \( 2 \leq i \leq \frac{n-1}{2} \) and \( 2 \leq j \leq \frac{n-1}{2} \), there exist one restricted value associated with \( v_6 \). Now we will move to the first copy of wheel. After labeling \( f(u_1) = b + 2 \) we assign \( b + 4 \) to v8 i.e. there exist one restricted value associated with \( u_1 \). Following in the similar way we can see that there exist one restricted value associated with each \( v_{2i+2} \) and \( u_{2j} - 1 \), where \( 2 \leq i \leq \frac{n-1}{2} \) and \( 2 \leq j \leq \frac{n-1}{2} \). Therefore, total number of restricted values associated with \( v_{2i+2} \) are and restricted values associated with \( u_{2j} - 1 \) are. Similarly we consider the pair of vertices \( (v_{2i-1}, u_{2j+2}) \), where \( 1 \leq i \leq 2 \) and \( 2 \leq j \leq \frac{n-1}{2} \). Applying the above procedure we can find the restricted values associated with \( v_{2i-1} \) and \( u_{2j+2} \). Therefore, total number of restricted values associated with \( v_{2i-1} \) are and restricted values associated with \( u_{2j+2} \) are
Since $d(u_{n-1}, z_1) = 3$, the radio condition becomes

\[ d(u_{n-1}, z_1) + |f(u_{n-1}) - f(z_1)| \geq 6, \text{ or } |f(u_{n-1}) - f(z_1)| \geq 3, \]

which implies that there exist two restricted values associated with $u_{n-1}$. If $z_1$ is labeled as $c$ (the value of $c$ must be greater than previously assigned integer), then any positive value from the set $\{c+1, c+2\}$ assigned to $z_2$ will not satisfy the radio condition which is defined above for the pair of vertices $(z_1, z_2)$. So, there are two restricted values associated with $z_1$.

Therefore, the total number of restricted values will be the sum of restricted values associated with $\{u_2, v_n, v_3, u_n, u_4\}$: restricted values associated with $v_{2i+2}$, restricted values associated with $u_{2j-1}$, restricted values associated with $u_{2j+2}$, restricted values associated with $u_{n-1}$, and restricted values associated with $z_1$.

Hence, $r_n(WH_n) \geq \text{allowed values} + \text{restricted values} = 2n + 2 + 2n + 5$, $r_n(WH_n) \geq 4n + 7$.

**Theorem 3.9.** For $n \geq 10$, $r_n(WH_n) \leq 4n + 7$.

**Proof.** We provide a radio labeling $f$ of $WH_n$ for $n \geq 10$. The span of this labeling will provide an upper bound for the radio number of $WH_n$. Starting with any copy of the Wheel subgraph of $WH_n$. Radio labeling $f : V(WH_n) \to \mathbb{Z}^+$ is defined as follows:

- When $n$ is even:
  
  \[
  \begin{align*}
  p(v_2) &= 1, \\
  p(v_3) &= 9, \\
  p(v_n) &= 5, \\
  p(v_1) &= 2n + 7, \\
  p(u_2) &= 2, \\
  p(u_3) &= 8, \\
  p(u_4) &= 16, \\
  p(u_1) &= 21, \\
  p(u_{n-1}) &= 12, \\
  p(v_{2i+2}) &= 15 + 4(i - 1), \\ 
  p(v_{2i+3}) &= (2n + 11) + 4(i - 1), \\
  p(u_{2i+3}) &= 25 + 4(i - 1), \\
  p(u_{2i+4}) &= (2n + 17) + 4(i - 1), \\
  p(z_1) &= 4n + 4, \\
  p(z_2) &= 4n + 7.
  \end{align*}
  \]

- When $n$ is even:
Examples of radio labeling (as defined in Theorem 3.9) for $n = 10$ and $n = 11$ are shown in Figure 4.

Claim: The labeling $f$ is a valid radio labeling i.e. the radio condition

$$d(u, v) + |f(u) - f(v)| \geq 1 + \text{diam}(WH_n) \geq 6$$

must hold for all distinct pairs of vertices of $WH_n$. We will discuss two cases for $n$.

When $n$ is even:
Case 1: In this case we consider the pairs of vertices \((v_i, v_j), (u_i, u_j)\), where \(1 \leq i, j \leq n\). Consider the pair \((v_i, v_j)\). As \(d(v_i, v_j) \leq 2\) and \(f(v_i) \in \{1, 5, 9, 15, \ldots, 2n+3, 2n+7, 2n+11, \ldots, 4n-1\}\). The possible label difference for each pair will satisfy \(|f(v_i) - f(v_j)| \geq 4\). So, the radio condition becomes

\[
d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 2 + 4, \quad \text{or} \quad d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 6.
\]

Hence, the radio condition is satisfied. Similarly we can check the radio condition for the pair of vertices \((u_i, u_j)\).

Case 2: In this case we consider the pairs of vertices \((z_1, z_2), (z_1, v_i), (z_1, u_i), (z_2, v_i)\) and \((z_2, u_i)\), where \(1 \leq i \leq n\). We will check the radio condition for \((z_1, z_2), (z_1, v_i)\) and \((z_1, u_i)\).

Subcase 2.1: Consider the pair \((z_1, z_2)\). Since \(d(z_1, z_2) = 3\), the radio condition becomes

\[
d(z_1, z_2) + |f(z_1) - f(z_2)| = 3 + |4n + 4 - 4n - 7| = 6.
\]

Subcase 2.2: Consider the pair \((z_1, v_i)\), where \(z_1 = v_i\). As \(d(z_1, v_i) = 1\) for \(1 \leq i \leq n\). Since \(f(v_i) \in \{1, 5, 9, 15, \ldots, 2n+3, 2n+7, 2n+11, \ldots, 4n-1\}\) and \(f(z_1) = 4n + 4\), the radio condition becomes

\[
d(z_1, v_i) + |f(z_1) - f(v_i)| \geq 1 + |4n + 4 - 4n + 1| = 6.
\]

Subcase 2.3: Consider the pair \((z_1, u_i)\), where \(z_1 = u_i\). As \(d(z_1, u_i) \leq 4\) for \(1 \leq i \leq n\). Since \(f(u_i) \in \{2, 8, 12, 16, 21, 25, \ldots, 2n+13, 2n+17, \ldots, 4n+1\}\) and \(f(z_1) = 4n + 4\), the radio condition becomes

\[
d(z_1, u_i) + |f(z_1) - f(u_i)| \geq 4 + |4n + 4 - 4n - 1|, \quad \text{or} \quad d(z_1, u_i) + |f(z_1) - f(u_i)| \geq 7,
\]

or

\[
d(z_1, u_i) + |f(z_1) - f(u_i)| \geq 6.
\]

Hence, the radio condition is satisfied in subcase 2.1, 2.2 and 2.3. Similarly we can check the radio condition for the pairs of vertices \((z_2, v_i)\) and \((z_2, u_i)\).

Case 3: Finally, consider \((v_i, u_j)\), where \(1 \leq i, j \leq n\). As \(d(v_i, u_j) \leq 5\) we have \(f(v_i) \in \{1, 5, 9, 15, 19, \ldots, 2n + 3, 2n + 7, 2n + 11, \ldots, 4n - 1\}\) and \(f(u_i) \in \{2, 8, 12, 16, 21, 25, \ldots, 2n + 13, 2n + 17, \ldots, 4n + 1\}\). The label difference for each pair will satisfy \(|f(v_i) - f(u_j)| \geq 1\). So, the radio condition becomes

\[
d(v_i, u_j) + |f(v_i) - f(u_j)| \geq 5 + 1 = 6.
\]

When \(n\) is odd:
Case 1: In this case we consider the pairs of vertices \((v_i, v_j)\) and \((u_i, u_j)\), where \(1 \leq i, j \leq n\).
Consider the pair \((v_i, v_j)\), where \(1 \leq i, j \leq n\). As \(d(v_i, v_j) \leq 2\) and \(f(v_i) \in \{1, 5, 9, 15, 19, \ldots, 2n+5, 2n+9, 2n+13, \ldots, 4n-1\}\). The possible label difference for each pair will satisfy \(|f(v_i) - f(v_j)| \geq 4\). So, the radio condition becomes
\[d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 2 + 4 = 6.\]
Hence, the radio condition is satisfied in this case. Similarly we can check the radio condition for the pair of vertices \((u_i, u_j)\) for \(1 \leq i, j \leq n\).

Case 2: In this case we consider the pairs of vertices \((z_l, z_2)\), \((z_l, v_i)\), \((z_l, u_i)\), \((z_2, v_i)\) and \((z_2, u_i)\), where \(1 \leq i \leq n\). We will check the radio condition for \((z_l, z_2)\), \((z_l, v_i)\) and \((z_l, u_i)\).

Subcase 2.1: Consider the pair \((z_l, z_2)\). Since \(f(z_l) = 4n + 4\), \(f(z_2) = 4n + 7\) and \(d(z_l, z_2) = 3\). So, the radio condition becomes
\[d(z_l, z_2) + |f(z_l) - f(z_2)| = 3 + |4n + 4 - 4n - 7| = 3 + 14n - 4 - 4n - 7 = 6.\]

Subcase 2.2: Consider the pair \((z_l, v_i)\), where \(z_l = v_i\). As \(d(z_l, v_i) = 1\) for \(1 \leq i \leq n\). Since \(f(v_i) \in \{1, 5, 9, 15, 19, \ldots, 2n+5, 2n+9, 2n+13, \ldots, 4n-1\}\) and \(f(z_l) = 4n + 4\). So, the radio condition becomes
\[d(z_l, v_i) + |f(z_l) - f(v_i)| \geq 1 + |4n + 4 - 4n + 1|, \text{ord}(z_l, v_i) + |f(z_l) - f(v_i)| \geq 1 + 5, \text{ord}(z_l, v_i) + |f(z_l) - f(v_i)| \geq 6.\]

Subcase 2.3: Consider the pair \((z_l, u_i)\), where \(z_l = u_i\). As \(d(z_l, u_i) \leq 4\) for \(1 \leq i \leq n\). Since \(f(u_i) \in \{2, 8, 12, 16, 21, 25, \ldots, 2n+11, 2n+15, \ldots, 4n+1\}\) and \(f(z_l) = 4n + 4\). So, the radio condition becomes
\[d(z_l, u_i) + |f(z_l) - f(u_i)| \geq 4 + |4n + 4 - 4n - 1|, \text{ord}(z_l, u_i) + |f(z_l) - f(u_i)| \geq 7, \text{ord}(z_l, u_i) + |f(z_l) - f(u_i)| \geq 6.\]
Hence, the radio condition is satisfied in subcase 2.1, 2.2 and 2.3. Similarly we can check the radio condition for the pairs of vertices \((z_2, v_i)\) and \((z_2, u_i)\).

Case 3: Finally, consider \((v_i, u_j)\) for \(1 \leq i, j \leq n\). As \(d(v_i, u_j) \leq 5\). We have \(f(v_i) \in \{1, 5, 9, 15, 19, \ldots, 2n + 5, 2n + 9, 2n + 13, \ldots, 4n - 1\}\) and \(f(u_i) \in \{2, 8, 12, 16, 21, 25, \ldots, 2n + 11, 2n + 15, \ldots, 4n + 1\}\). The possible difference of labels for each pair will satisfy \(|f(v_i) - f(u_j)| \geq 1\). The radiocondition becomes
\[d(v_i, u_j) + |f(v_i) - f(u_j)| \geq 5 + 1, \text{ord}(v_i, u_j) - |f(v_i) - f(u_j)| \geq 6.\]
Hence, the radio condition is satisfied in this case.
These three cases (for n is even and odd) establish the claim that \( f \) is a valid radio labeling of \( WH_n \). Thus, \( rn(WH_n) \leq \text{span}(f) = 4n + 7 \).

Note: For \( n = 3 \), diameter of \( WH_3 \) is 3. It is easy to find that the radio number of \( WH_3 \) is 12. For \( 4 \leq n \leq 9 \), the \( rn(WH_n) \) cannot be found using the above procedure. It is easy to see that for \( 3 \leq n \leq 9 \), we have

**Theorem 3.10.** For \( n \geq 10 \) the radio number of Joint Wheel graph \( (WH_n) \) is

\[
   rn(WH_n) = 4n + 7.
\]

**Proof.** Follows from Theorem 3.8 and Theorem 3.9.

**Open problem:** Investigate the \( rn(FW^k_n) \) when the copies of Wheel graph has different number of vertices.

**References.**


