ONE MODULO N GRACEFULNESS OF REGULAR BAMBOO TREE AND COCONUT TREE

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Abstract

A function \( f \) is called a graceful labelling of a graph \( G \) with \( q \) edges if \( f \) is an injection from the vertices of \( G \) to the set \( \{0, 1, 2, \ldots, q\} \) such that, when each edge \( xy \) is assigned the label \( |f(x) - f(y)| \), the resulting edge labels are distinct. A graph \( G \) is said to be one modulo \( N \) graceful (where \( N \) is a positive integer) if there is a function \( \varphi \) from the vertex set of \( G \) to \( \{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\} \) in such a way that (i) \( \varphi \) is 1 − 1 (ii) \( \varphi \) induces a bijection \( \varphi_\_ \) from the edge set of \( G \) to \( \{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\} \) where \( \varphi_\_(uv)=|\varphi(u) - \varphi(v)| \). In this paper we prove that the every regular bamboo tree and coconut tree are one modulo \( N \) graceful for all positive integers \( N \).

1. INTRODUCTION

S.W.Golomb [2] introduced graceful labelling. Odd gracefulness was introduced by B.Gnanajothi [1] . C.Sekar [6] introduced one modulo three graceful labelling. V.Ramachandran and C.Sekar [4] introduced the concept of one modulo \( N \) graceful where \( N \) is any positive integer. In the case \( N = 2 \), the labelling is odd graceful and in the case \( N = 1 \) the labelling is graceful. [6] Every regular bamboo tree is graceful. In this paper we establish the result for one modulo \( N \) graceful (\( N > 1 \) ) of the regular bamboo tree and also we prove that coconut tree is one modulo \( N \) graceful for all positive integers \( N \). In order to prove the existing conjecture

Problem 1. All trees are graceful?
Problem 2. All lobsters are graceful?

we take a diversion to prove one modulo \( N \) graceful of acyclic graphs. Sometimes the technique involved in one modulo \( N \) graceful labelling may yield a new approach to have graceful labelling of graphs. Our approach will motivate the scholars to do more research in this area.
**Main Results**

**Definition 2.1.** A graph $G$ with $q$ edges is said to be one modulo $N$ graceful (where $N$ is a positive integer) if there is a function $\phi$ from the vertex set of $G$ to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$ in such a way that (i) $\phi$ is 1-1 (ii) $\phi$ induces a bijection $\phi_-$ from the edge set of $G$ to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$ where $\phi_-(uv) = |\phi(u) - \phi(v)|$.

**Definition 2.2.** Consider $k$ copies of paths $P_n$ of length $n-1$ and stars $S_m$ with $m$ pendant vertices. Identify one of the two pendant vertices of the $j$th path with the centre of the $j$th star. Identify the other pendant vertex of each path with a single vertex $u_0$ ( $u_0$ is not in any of the star and path). The graph obtained is a regular bamboo tree. **Definition 2.3.** A coconut Tree $CT(m, n)$ is the graph obtained from the path $P_n$ by appending $m$ new pendant edges at an end vertex of $P_n$.

**Theorem 2.4.** Every regular bamboo tree is one modulo $N$ graceful for every positive integer $N > 1$.

**Proof:** Let $v_1^{(j)}, v_2^{(j)}, \ldots, v_{n-1}^{(j)}$ be the vertices of the $j$th path where $v_1^{(j)}$ is identified with $u_0$ and $v_{n-1}^{(j)}$ is identified with $v_0^{(j)}$ which is the centre of the $j$th star. Let $v_1^{(j)}, v_2^{(j)}, \ldots, v_{m-1}^{(j)}$ be the pendant vertices of the $j$th star. The bamboo tree has $k(n+m-1)+1$ vertices and $k(n+m-1)$ edges. Name the vertices as shown in the figure.

![Diagram](image)

**Case (i) $k$ is odd and $n$ is odd**

Define
We compute the edge labelling in the following sequence.

For $1 \leq j \leq k$

$$| \phi(u_2^{(j)}) - \phi(u_0) | = Nk(n + m - 1) - Nj + 1$$

For $1 \leq r \leq m$ and $j = 1, 2, \ldots, \frac{k-1}{2}$

$$| \phi(v_r^{(j)}) - \phi(u_n^{(j)}) | = Nk(m - r + 1) - 2Nj + 1$$

For $1 \leq r \leq m$ and $j = \frac{k+1}{2}, \frac{k+3}{2}, \ldots, k$

$$| \phi(v_r^{(j)}) - \phi(u_n^{(j)}) | = Nk(m - r + 2) - 2Nj + 1$$

For $j = 2, 4, \ldots, n - 1$ and $j = 1, 2, \ldots, \frac{k-1}{2}$

$$| \phi(u_i^{(j)}) - \phi(u_{i+1}^{(j)}) | = Nk(n + m - i) - 2Nj + 1$$

For $j = 2, 4, \ldots, n - 1$ and $j = \frac{k+1}{2}, \frac{k+3}{2}, \ldots, k$

$$| \phi(u_i^{(j)}) - \phi(u_{i+1}^{(j)}) | = Nk(n + m - i + 1) - 2Nj + 1$$

This shows that the edges have the distinct labels $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}.$
Example 2.5. One modulo 5 graceful labelling of regular bamboo tree. ( k = 5 , n = 5 , m = 3 )

Case (ii) k is odd and n is even
Define

\[ \phi : \{1, N+1, 2N+1, \ldots, N(q-1)+1\} \rightarrow \{0, 1, 2, \ldots, N\} \]

Hence the regular bamboo tree is one modulo N graceful.
Clearly \( \phi \) defines a one modulo N graceful labelling of regular bamboo tree.
Case (iii) $k$ is even and $n$ is odd

Define

$$\phi(u_0) = \phi(u_1) = 0$$

For $i = 2, 4, \ldots, n$

$$\phi(u_i) = Nk(n + m - 1) - (N - 1) - N(j - 1) - \frac{Nk(i-2)}{2}$$

for $j = 1, 2, \ldots, k$

For $i = 3, 5, \ldots, n - 1$

$$\phi(u_i) = \left\{ \begin{array}{ll}
N(k + 1) + N(j - 1) + \frac{Nk(i-1)}{2} & \text{for } j = 1, 2, \ldots, \frac{(k-1)}{2} \\
N(k + 1) + N(j - \frac{(k+1)}{2}) + \frac{Nk(i-3)}{2} & \text{for } j = \frac{(k+1)}{2}, \frac{(k+3)}{2}, \ldots, k
\end{array} \right.$$
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\( \phi(v_{i}^{(1)}) = N(r - 1) \) for \( r = 1, 2, \ldots, m \)

\[ \phi(u_{0}) = NK(n + m - 1) - (N - 1) - \frac{N(n-1)}{2} \]

\[ \phi(u_{i}^{(1)}) = \begin{cases} 
N(m - 1) + \frac{N(n-1)}{2} + \frac{N(k-1)(k-2)}{2} & \text{for } i = 2, 4, \ldots, n-1 \\
NK(m + n - 1) + 1 - \frac{N(n-1)}{2} + \frac{N(k-1)(k-2)}{2} & \text{for } i = 3, 5, \ldots, n
\end{cases} \]

\[ \phi(u_{i}^{(2)}) = NK(n + m - 1) \cdot N(j - 2) \cdot N(n-1+k) \cdot N(k-1)(k-3) \quad \text{for } j = 2, 3, \ldots, k \]

For \( i = 2, 4, \ldots, n - 1 \)

\[ \phi(u_{i}^{(3)}) = \begin{cases} 
N(m + \frac{N(n-1+k)}{2}) + N(j - 2) + \frac{N(k-1)(k-2)}{2} & \text{for } j = 2, 3, \ldots, k \\
NK(m - 1) + \frac{N(n-1)}{2} + N + N(j - \frac{k}{2} - 1) + \frac{N(k-1)(k-2)}{2} & \text{for } j = \frac{k}{2} + 1, \frac{k}{2} + 2, \ldots, k
\end{cases} \]

For \( r = 1, 2, \ldots, m \)

\[ \phi(v_{r}^{(1)}) = \begin{cases} 
N(m + \frac{N(n-1+k)}{2}) + N(j - 2) + \frac{N(k-1)(k-2)}{2} & \text{for } j = 2, 3, \ldots, k \\
NK(m - 1) + \frac{N(n-1)}{2} + N(k - \frac{k}{2} - 1) + \frac{N(k-1)(k-2)}{2} & \text{for } j = \frac{k}{2} + 1, \frac{k}{2} + 2, \ldots, k
\end{cases} \]

The proof is similar to the proof in case(i).
Clearly \( \phi \) defines a one modulo \( N \) graceful labelling of regular bamboo tree.

Example 2.7. One modulo 5 graceful labelling of regular bamboo tree. ( \( k = 6 \), \( n = 7 \), \( m = 3 \) )
The proof is similar to the proof in case(i).

Example 2.8. One modulo $10$ graceful labelling of regular bamboo tree. ($k = 8$, $n = 6$, $m = 2$)
Thus it is clear that the vertices have distinct labels. Therefore $\phi$ is 1 − 1.

We compute the edge labelling in the following sequence.

For $1 \leq i \leq m$
\[ |\phi(u_{2i}) - \phi(u_{2i - 1})| = N(2k + i - 1) + 1 \]

For $1 \leq i \leq k$
\[ |\phi(u_{2i}) - \phi(u_{2i - 1})| = N(2k + 1 - 2i) + 1 \]
\[ |\phi(u_{2i}) - \phi(u_{2i + 1})| = N(2k - 2i) + 1 \]

This shows that the edges have the distinct labels \( \{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\} \).

It is clear from the above labelling that the function $\phi$ from the vertex set of $G$ to \( \{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\} \) is in such a way that (i) $\phi$ is 1 − 1 (ii) $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to \( \{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\} \) where $\phi^*(uv) = |\phi(u) - \phi(v)|$.

Hence the coconut tree is one modulo $N$ graceful. Clearly $\phi$ defines a one modulo $N$ graceful labelling of coconut tree.

Example 2.10. One modulo 3 graceful labelling and graceful labelling of coconut tree.
Case (ii) $n$ is even.
Let $n = 2k$.
Define
\[
\phi(u_{2i-1}) = 2Nk - (2N - 1) - N(i - 1) \quad \text{for } i = 1, 2, 3, \ldots, k
\]
\[
\phi(v_i) = 2Nk + 1 + N(i - 1) \quad \text{for } i = 1, 2, 3, \ldots, m
\]
The proof is similar to the proof in case (i).

Clearly $\phi$ defines a one modulo $N$ graceful labelling of coconut tree.

**Example 2.11.** One modulo 10 graceful labelling and odd graceful labelling of coconut tree.
References