NONLINEAR OBSERVER DESIGN FOR L-V SYSTEM

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ABSTRACT

This paper investigates the exponential observer problem for the Lotka-Volterra (L-V) systems. We have applied Sundarapandian's theorem (2002) for nonlinear observer design to solve the problem of observer design problem for the L-V systems. The results obtained in this paper are applicable to solve the problem of nonlinear observer design problem for the L-V models of population ecology in the food webs. Two species L-V predator-prey models are studied and nonlinear observers are constructed by applying the design technique prescribed in Sundarapandian's theorem (2002). Numerical simulations are provided to describe the nonlinear observers for the L-V systems.

KEYWORDS

Lotka-Volterra Models, Nonlinear Observers, Exponential Observers, Observability, Ecosystems.

1. INTRODUCTION

Nonlinear observer design is one of the central problems in the control systems literature. An observer for a control system is an estimator of the state of the system, which is very useful for implementing feedback stabilization or feedback regulation of nonlinear control systems.

The problem of designing observers for linear control systems was first introduced and solved by Luenberger ([1], 1966). The problem of designing observers for nonlinear control systems was first proposed by Thau ([2], 1973). There has been significant research done on the nonlinear observer design problem over the past three decades [2-12].

A necessary condition for the existence of an exponential observer for nonlinear control systems was obtained by Xia and Gao ([3], 1988). On the other hand, sufficient conditions for nonlinear observer design have been obtained in the control systems literature from an impressive variety of points of view. Kou, Elliott and Tarn ([4], 1975) obtained sufficient conditions for the existence of exponential observers using Lyapunov-like method.

In [5-8], suitable coordinates transformations were found under which a nonlinear control system is transformed into a canonical form, where the observer design is carried out. In [9], Tsinias derived sufficient Lyapunov-like conditions for the existence of asymptotic observers for nonlinear systems. A harmonic analysis approach was proposed by Celle et. al. ([10], 1989) for the synthesis of nonlinear observers.

A complete characterization of local exponential observers for nonlinear control systems was first obtained by Sundarapandian ([11], 2002). Sundarapandian's theorem (2002) for nonlinear observer design was proved using Lyapunov stability theory. In [11], necessary and sufficient conditions were obtained for exponential observers for Lyapunov stable nonlinear systems and an exponential observer design was provided which generalizes the linear observer design of Luenberger [1]. Krener and Kang ([12], 2003) introduced a new method for the design of nonlinear observers for nonlinear control systems using backstepping.

Lotka-Volterra (L-V) system is an important interactive model of nonlinear systems, which was discovered independently by the Italian mathematician Vito Volterra ([13], 1926) and the American biophysicist Alfred Lotka ([14], 1925). Recently, there has been significant interest in the application of mathematical systems theory to population biology systems [15-16]. A survey paper by Varga ([15], 2008) reviews the research done in this area.

This paper is organized as follows. In Section 2, we review the concept and results of exponential observers for nonlinear systems. In Section 3, we detail the design of nonlinear observers for two species L-V systems. In Section 4, we provide numerical examples for the nonlinear observer design for L-V systems.

2. NONLINEAR OBSERVER DESIGN

By the concept of a *state observer* for a nonlinear system, it is meant that from the observation of certain states of the system considered as outputs, it is desired to estimate the state of the whole system as a function of time. Mathematically, we can define nonlinear observers as follows.

Consider the nonlinear system described by

$$\dot{x} = f(x) \tag{1a}$$

$$y = h(x) \tag{1b}$$

where $x \in R^n$ is the state and $y \in R^p$ the output.

It is assumed that $f : R^n \to R^n$, $h : R^n \to R^p$ are C^1 mappings and for some $x^* \in R^n$, the following hold:

$$f(x^*) = 0, \; h(x^*) = 0.$$

Note that the solutions x^* of $f(x) = 0$ are called the *equilibrium points* of the dynamics (1a).

The linearization of the nonlinear system (1) at the equilibrium x^* is given by

$$\dot{x} = Ax \tag{2a}$$

$$y = Cx \tag{2b}$$

where

$$A = \left[\frac{\partial f}{\partial x} \right]_{x = x^*}$$

and

$$C = \left[\frac{\partial h}{\partial x} \right]_{x = x^*}.$$

Next, the definition of nonlinear observers for the nonlinear system (1) is given as stated in [15].

Definition 1. (Sundarapandian, [11], 2002)

A C^1 dynamical system described by

$$\dot{z} = g(z, y), \qquad (z \in R^n) \tag{3}$$

is called a **local asymptotic** (respectively, **exponential**) observer for the nonlinear system (1) if the composite system (1) and (3) satisfies the following two requirements:

(O1) If $z(0) = x(0)$, then $z(t) = x(t), \quad \forall t \geq 0.$

(O2) There exists a neighbourhood V of the equilibrium x^* of R^n such that for all $z(0), x(0) \in V$, the estimation error $e(t) = z(t) - x(t)$ decays *asymptotically* (resp. *exponentially*) to zero.

Theorem 1. (Sundarapandian, [11], 2002)

Suppose that the nonlinear system (1) is Lyapunov stable at the equilibrium x^* and that there exists a matrix K such that $A - KC$ is Hurwitz. Then the dynamical system defined by

$$\dot{z} = f(z) + K[y - h(z)] \tag{4}$$

is a local exponential observer for the nonlinear system (1).

Remark 1. If the estimation error is defined as $e = z - x$, then the *estimation error* is governed by

$$\dot{e} = f(e + x) - f(x) - K[h(x + e) - h(x)] \tag{5}$$

Linearizing the dynamics (5) at x^* yields the system

$$\dot{e} = Ee, \text{ where } E = A - KC. \tag{6}$$

If (C, A) is observable, then the eigenvalues of $E = A - KC$ can be arbitrarily placed in the complex plane and thus, a local exponential observer of the form (4) can be always found so that the transient response of the error decays fast with any desired speed of convergence.

3. NONLINEAR OBSERVER DESIGN FOR TWO-SPECIES L-V SYSTEMS

The two species *Lotka-Volterra* (L-V) system consists of the following system of differential equations

$$\begin{aligned} \dot{x}_1 &= -ax_1 + bx_1x_2 \\ \dot{x}_2 &= -cx_1x_2 + \frac{r}{M}x_2(M - x_2) \end{aligned} \tag{7}$$

In the system (7), $x_1(t)$ and $x_2(t)$ represent the predator and prey population densities respectively as a function of time.

The parameters a, b, c, r and M are all positive, where a represents the natural decay rate of the predator population in the absence of prey, r represents the natural growth rate of prey population in the absence of predators, b represents the efficiency and propagation rate of the predator in the presence of prey, c represents the effect of predation on the prey and M is the carrying capacity.

The equilibrium points of the L-V system (7) are obtained by setting $\dot{x}_1 = 0,\ \dot{x}_2 = 0$ and solving the resulting nonlinear equations for x_1 and x_2.

A simple calculation shows that the system (7) has three equilibrium points, *viz.*

$$(0,0), \quad (0, M) \quad \text{and} \quad x^* = \left(x_1^*, x_2^*\right) = \left[\frac{r}{Mc}\left(M - \frac{a}{b}\right), \frac{a}{b}\right].$$

Using Lyapunov stability theory, it can be easily shown that the equilibrium points $(0,0)$ and $(0, M)$ are unstable, while the equilibrium point $x = x^*$ is asymptotically stable.

Figure 1 depicts the state orbits of the L-V system (7) when $a = b = c = r = 1$ and $M = 400$. This figure shows that the equilibrium $x = x^*$ is locally asymptotically stable.

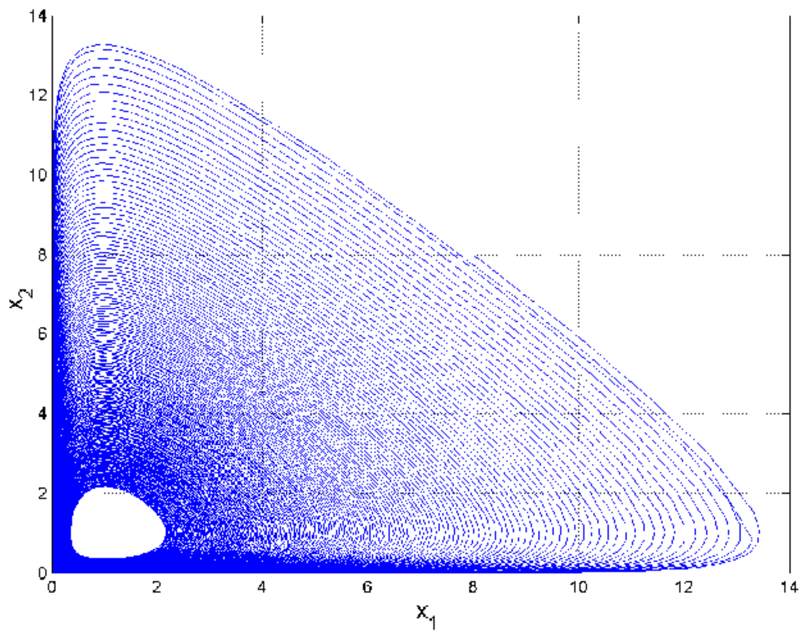

Figure1. State Orbits of the L-V System

Since only the stable equilibrium $x^* = (x_1^*, x_2^*)$ pertains to problems of practical interest, we consider only the problem of nonlinear observer design for the two species LV system (7) near $x = x^*$. Since the carrying capacity M is large, it is evident that the coordinates of the equilibrium $x^* = (x_1^*, x_2^*)$ are positive.

Next, we suppose that the prey population is given as the output function of the LV system (7). Since we know the value of x_2^*, it is convenient to take the output function as

$$y = x_2 - x_2^* \qquad (8)$$

The linearization of the plant dynamics (7) at $x = x^*$ is given by

$$A = \begin{bmatrix} 0 & bx_1^* \\ -cx_2^* & -\dfrac{r}{M}x_2^* \end{bmatrix} \qquad (9)$$

The linearization of the output function (8) at $x = x^*$ is given by

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad (10)$$

From (9) and (10), the observability matrix for the L-V system (7) with output (8) is found as

$$\boldsymbol{O}(C, A) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -cx_2^* & -\dfrac{r}{M}x_2^* \end{bmatrix}$$

which has full rank. This shows that the system pair (C, A) is completely observable.

Also, we have shown that the equilibrium $x = x^*$ of the L-V system (7) is locally asymptotically stable.

Thus, by Sundarapandian's result (Theorem 1), we obtain the following result, which gives an useful formula for the construction of nonlinear observer for the two species L-V system.

Theorem 2. The two species L-V system (7) with the output function (8) has a local exponential observer of the form

$$\dot{z} = f(z) + K\left[y - h(z)\right], \qquad (11)2$$

where K is a matrix chosen such that $A - KC$ is Hurwitz. Since (C, A) is observable, a gain matrix K can be chosen so that the error matrix $E = A - KC$ has arbitrarily assigned set of eigenvalues with negative real parts.

Example 1. Consider the two-species L-V system (7), where $a = 0.1, b = 0.3, c = 0.7, r = 10$ and $M = 100.$ Thus, we consider the L-V system as

$$\begin{aligned} \dot{x}_1 &= -0.1x_1 + 0.3x_1x_2 \\ \dot{x}_2 &= 10x_2 - 0.7x_1x_2 - 0.1x_2^2 \end{aligned} \qquad (12)$$

The system (12) has the stable equilibrium given by

$$x^* = (x_1^*, x_2^*) = (14.2281, 0.3333)$$

We consider the output function as

$$y = x_2 - x_2^* = x_2 - 0.3333 \qquad (13)$$

The linearization of the system (12)-(13) at $x = x^*$ is obtained as

$$A = \begin{bmatrix} 0 & 4.2714 \\ -0.2333 & -0.0333 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

As we have already shown for the general case, the system pair (C, A) is observable. Thus, the eigenvalues of the error matrix $E = A - KC$ can be arbitrarily placed.

Using the Ackermann's formula for the observability gain matrix [24], we can choose K so that the error matrix $E = A - KC$ has the eigenvalues $\{-4, -4\}$. A simple calculation gives

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -64.3000 \\ 7.9667 \end{bmatrix}$$

By Theorem 2, a local exponential observer for the given two species L-V model (12)-(13) around the positive equilibrium $(x_1^*, x_2^*) = (14.2281, 0.3333)$ is given by

$$\begin{aligned} \dot{z}_1 &= -0.1z_1 + 0.3z_1z_2 + k_1(y - z_2 + 0.3333) \\ \dot{z}_2 &= 10z_2 - 0.7z_1z_2 - 0.1z_2^2 + k_2(y - z_2 + 0.3333) \end{aligned} \tag{14}$$

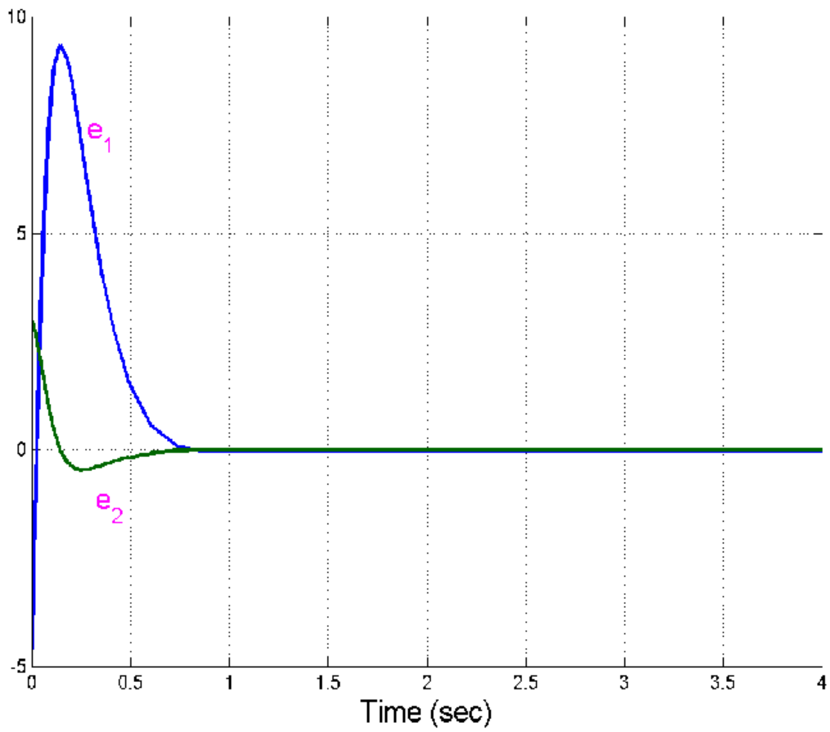

Figure 2. Time History of the Estimation Error

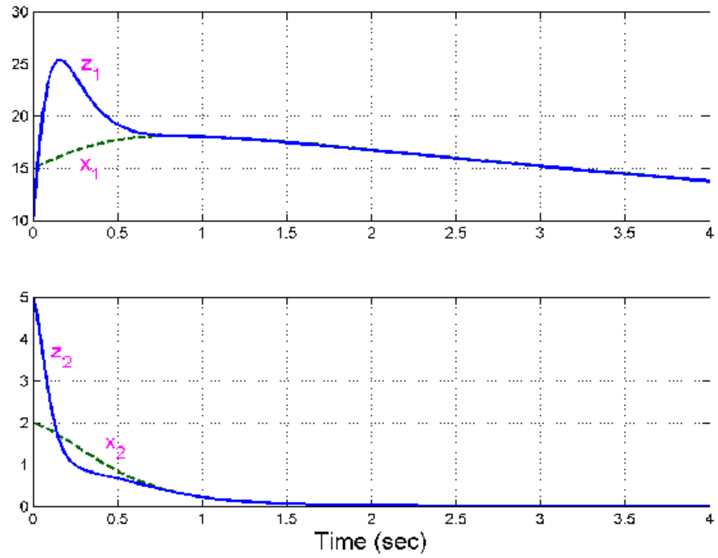

Figure 3. Local Exponential Observer for the L-V System

For numerical simulation, we take the initial conditions as

$$x(0) = \begin{bmatrix} 15 \\ 2 \end{bmatrix} \quad \text{and} \quad z(0) = \begin{bmatrix} 10 \\ 5 \end{bmatrix}.$$

Figure 2 depicts the time history of the error estimates $e_1 = z_1 - x_1$ and $e_2 = z_2 - x_2$.

Figure 3 depicts the synchronization of the states x_1 and x_2 of the L-V system (12) with the states z_1 and z_2 of the observer system (14).

3. CONCLUSIONS

For many real problems of population ecosystems, an efficient monitoring system is of great importance. In this paper, the methodology based on Sundarapandian's theorem (2002) for nonlinear observer design is suggested for the monitoring of two-species Lotka-Volterra (L-V) population ecology systems. The results have been proved using Lyapunov stability theory. Under the condition of stable coexistence of all the species, an exponential observer is constructed near a non-trivial equilibrium of the Lotka-Volterra population ecology system using Sundarapandian's theorem (2002). A numerical example has been worked out in detail for the local exponential observer design for the 2-species Lotka-Volterra population ecology systems.

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