HOSOYA POLYNOMIAL, WIENER AND HYPER-WIENER INDICES OF SOME REGULAR GRAPHS

Mohammad Reza Farahani

Department of Applied Mathematics, Iran University of Science and Technology (IUST)
Narmak, Tehran, Iran

ABSTRACT

Let G be a graph. The distance d(u,v) between two vertices u and v of G is equal to the length of a shortest path that connects u and v. The Wiener index W(G) is the sum of all distances between vertices of G, whereas the hyper-Wiener index WW(G) is defined as WW(G) = \( \sum_{\{u,v\} \subseteq V(G)} \left( d(u,v) + d(v,u)^2 \right) \). Also, the Hosoya polynomial was introduced by H. Hosoya and define H(G,x) = \( \sum_{\{u,v\} \subseteq V(G)} x^{d(u,v)} \). In this paper, the Hosoya polynomial, Wiener index and Hyper-Wiener index of some regular graphs are determined.

KEYWORDS

Network Protocols, Topological distance, Hosoya polynomial, Wiener Index, Hyper-Wiener index, Regular graph, Harary graph.

1. INTRODUCTION

Let G=(V;E) be a simple connected graph. The sets of vertices and edges of G are denoted by V=V(G) and E=E(G), respectively. The distance between vertices u and v of G, denoted by d(u,v), is the number of edges in a shortest path connecting them. An edge e=uv of graph G is joined between two vertices u and v (d(u,v)=1). The number of vertex pairs at unit distance equals the number of edges. Also, the topological diameter D(G) is the longest topological distance in a graph G.

A topological index of a graph is a number related to that graph which is invariant under graph automorphism. The Wiener index W(G) is the oldest topological indices, (based structure descriptors) [8], which have many applications and mathematical properties and defined by Harold Wiener in 1947 as:

\[
W(G) = \frac{1}{2} \sum_{v \in V(G)} \sum_{u \in V(G)} d(v,u)
\]  

(1.1)

Also, for this topological index, there is Hosoya Polynomial. The Hosoya polynomial was introduced by H. Hosoya, in 1988 [3] and define as follow:

\[
H(G,x) = \frac{1}{2} \sum_{v \in V(G)} \sum_{u \in V(G)} x^{d(v,u)}
\]  

(1.2)
The Hosoya polynomial and Wiener index of some graphs computed [1, 2-4, 5]. Another topological index of graph (based structure descriptor) that was conceived somewhat later is the hyper-Wiener index that introduced by Milan Randić in 1993 [6] as

\[
WW(G) = \frac{1}{2} \sum_{v \in V(G)} \sum_{u \in V(G)} \left( d(v,u) + d(v,u)^2 \right)
\]

(1.3)

\[
= \frac{1}{2} W(G) + \frac{1}{2} \sum_{v \in V(G)} \sum_{u \in V(G)} d(v,u)^2
\]

(1.4)

In this paper, we obtained a closed formula of the Hosoya polynomial, Wiener and Hyper-Wiener indices for an interesting regular graph that called Harary graph. The general form of the Harary graph \( H_{2m,n} \) is defined as follows:

**Definition 1.1** [7]. Let \( m \) and \( n \) be two positive integer numbers, then the Harary graph \( H_{2m,n} \) is constructed as follows:

It has vertices \( 1, 2, ..., n-1, n \) and two vertices \( i \) and \( j \) are joined if \( i-m \leq j \leq i+m \) (where addition is taken modulo \( n \)).

![Figure 1. The Harary graph \( H_{6,10} \).](image)

**2. MAIN RESULTS**

In this section we compute the Hosoya polynomial, Wiener index and hyper-Wiener index of the Harary graph \( H_{2m,n} \).

**Theorem 2.1.** Consider the Harary graph \( H_{2m,n} \) for all positive integer number \( m \) and \( n \). Then,

- The Hosoya polynomial of \( H_{2m,n} \) for \( n \) odd is

\[
H_{2m,n} = \sum_{d=1}^{[\gamma_{2m}]} mnx^d + n \left( \left[ \frac{n}{2} \right] - m \times \left[ \frac{n}{2m} \right] \right) x^{[\gamma_{2m}]+1}
\]

- The Hosoya polynomial of \( H_{2m,2q} \) for \( n \) even (\( = 2q \)) is

\[
H_{2m,2q} = \sum_{d=1}^{[\gamma]} 2qmx^d + 2q^2 - 2qm \left[ \frac{q}{m} \right] - q \times \left[ \frac{q}{m} \right] + 1
\]
Let \( H \) be the Harary graph \( H_{2m,n} \) (Figure 1). The number of vertices in this graph is equal to \( |V(H_{2m,n})| = n \). Also, two vertices \( v_i, v_j \in V(H_{2m,n}) \) are adjacent if and only if \( |i-j| \leq m \), then \( d(v_i, v_j) = 1 \) thus the number of edges of this regular graph is \( |E(H_{2m,n})| = \frac{n(2m)}{2} = mn \).

It is obvious that for all graph \( G \), \( d(G, 1) = \sum_{i=1}^{d(G)} d(G, i) = \binom{n}{2} = \frac{n(n-1)}{2} \). On the other hand from the structure of \( H_{2m,n} \) (Figure 1), we see that all vertices of \( H_{2m,n} \) have similar geometrical and topological conditions then the number of path as distance \( i \) in \( H_{2m,n} \) (\( d(H_{2m,n}, i) \)) is a multiple of the number of vertices \( (n) \). In other words, any vertex \( v \in V(H_{2m,n}) \) is an endpoint of \( d(H_{2m,n}, i)/n \) paths as distance \( i \) in \( H_{2m,n} \) \( (=d(H_{2m,n}, i)) \). For example \( \forall v \in V(H_{2m,n}) \), there are \( 2m = d(H_{2m,n}, 1) \) paths as distance \( 1 \) (or all edges incident to \( v \)) in \( H_{2m,n} \). From Figure 1, one can see that \( \forall i=1,2,\ldots,n \) and \( v_i, v_{i+m} \in V(H_{2m,n}) \), \( d(v_i, v_{i+m}) = 2 \) \( (j = 1,2,\ldots,m) \) because \( v_i, v_{i+m}, v_{i+m}, v_{i+m} \in E(H_{2m,n}) \) and \( d(v_i, v_{i+m}) = d(v_{i+m}, v_{i+m} + 1) = 1 \). Obviously \( d_i(H_{2m,n}, 2) = 2m \).

Now, since \( \forall i \in \{1,2,\ldots,n\} \) and \( v_i, v_{i+m} \in V(H_{2m,n}) \), \( d(v_i, v_{i+m}) = d(v_{i+m}, v_{i+2m}) = d(v_{i+2m}, v_{i+3m}) = \ldots = d(v_{i+(m-1)m}, v_{i+km}) = 1 \) such that \( k \leq n/2m \). Thus \( \forall d = 1,2,\ldots,k \) \( d(v_i, v_{i+dm}) = d \) and obviously \( d(v_i, v_{i+dm+j}) = d+1 \) \( (j = 1,2,\ldots,m-1) \). By these mentions, one can see that for an arbitrary vertex \( v_i \) the diameter \( D(H_{2m,n}) \) of this Harary graph is \( d(v_i, v_{i+[n/2]}) = \lfloor n/2m \rfloor + 1 \). Also the distance between vertices \( v_i \) and \( v_{i+(n/2m)+j} \) is equal to \( D(H_{2m,n}) \) for all \( j = 1,2,\ldots,[n/2] \). It's easy to see that if \( n \) be odd then \( \forall v \in V(H_{2m,n}) \), \( d_i(H_{2m,n}, 1) = \left\lfloor n/2m \right\rfloor - m \left\lfloor n/2m \right\rfloor \) for \( 1 \leq i \leq \left\lfloor n/2m \right\rfloor \), else \( n \) be even then \( d_i(H_{2m,n}, 1) = \left\lfloor n/2m \right\rfloor - m \left\lfloor n/2m \right\rfloor \).

Now by using the definition of Hosoya polynomial (equation 2), we have \( H(H_{2m,n}) = \sum_{i=1}^{n} x^{d(v_i, v_{i+1})} = \sum_{i=1}^{n} x^d(v_i, v_{i+1}) \) and \( \sum_{i=1}^{n} x^{d(v_i, v_{i+1})} = \sum_{i=1}^{n} \sum_{j=0}^{m-1} \sum_{k=0}^{\left\lfloor n/2m \right\rfloor} x^{d(v_i, v_{i+1})} \) and \( d(H_{2m,n}, 1) = \left\lfloor n/2m \right\rfloor + \left\lfloor n/2m \right\rfloor - \left\lfloor n/2m \right\rfloor \). Here the proof of Theorem 2.1 is completed.

Theorem 2.2. Let \( H_{2m,n} \) be the Harary graph. Then the Wiener index of \( H_{2m,n} \) is equal to

- If \( n \) be odd, \( W(H_{2m,2q+1}) = (2q + 1) \left( q + \left\lfloor q/m \right\rfloor \left( q - \frac{m}{2} \right) - \frac{m}{2} \left\lfloor q/m \right\rfloor \right) \)

- If \( n \) be even, \( W(H_{2m,2q}) = (2q^2 - q) + q \left\lfloor q/m \right\rfloor (2q - m) - qm \left\lfloor q/m \right\rfloor \)
**Proof.** By according to the definitions of the Wiener index and Hosoya Polynomial of a graph $G$ (Equation 1.1 and 1.2), then the Wiener index $W(G)$ will be the first derivative of Hosoya polynomial $H(G,x)$ evaluated at $x=1$. Thus by using the results from Theorem 2.1, we have

1- If $n$ be an arbitrary even positive integer number ($n=2q$), then

$$W(H_{2m,2q}) = \sum_{(u,v) \in V(H_{2m,2q})} d(u,v)$$

$$= \frac{\partial H(H_{2m,2q},x)}{\partial x}_{x=1}$$

$$= \frac{\partial}{\partial x} \left[ \sum_{d=1}^{\lfloor \sqrt{2q} \rfloor} 2qm^d + 2q^2 - 2qm \left( \frac{q}{m} \right) - q \left( \frac{q}{m} \right)^2 \right]_{x=1}$$

$$= 2qm \sum_{d=1}^{\lfloor \sqrt{2q} \rfloor} d + \left( -2qm \left( \frac{q}{m} \right)^2 + \left( \frac{q}{m} \right)^2 \left( 2q^2 - q - 2qm \right) + (2q^2 - q) \right)$$

$$= (2q^2 - q) + q(2q - 1 - m) \left( \frac{q}{m} \right) - q \left( \frac{q}{m} \right)^2$$

2- If $n$ be an odd positive integer number ($n=2q+1$), then

$$W(H_{2m,2q+1}) = \frac{\partial}{\partial x} \left[ \sum_{d=1}^{\lfloor \sqrt{2q} \rfloor} mn^d + n \left( \left\lfloor \frac{n}{2} \right\rfloor - m \left\lfloor \frac{n}{2m} \right\rfloor \right) \left( \frac{q}{m} \right) \left( \frac{q}{m} \right)^2 \right]_{x=1}$$

$$= (2q + 1) \left( \sum_{d=1}^{\lfloor \sqrt{2q} \rfloor} m \cdot d + \left( q - m \left( \frac{q}{m} \right) \left( \frac{q}{m} \right)^2 + 1 \right) \right)$$

$$= (2q^2 + q) + \left( \frac{q}{m} \right) \left( 2q^2 + q - m \left( \frac{2q + 1}{2} \right) - m \left( \frac{2q + 1}{2} \right) \left( \frac{q}{m} \right)^2 \right).$$

**Corollary 2.3.** Suppose $n$ be an even positive integer number ($n=2q$) and $2mn$, then such that $q/m=r$. Thus $W(H_{2m,2m})=mr(m^2+mr-r-1)$.

**Theorem 2.4.** The hyper-Wiener index of the Harary graphs $H_{2m,2q}$ and $H_{2m,2q+1}$ are equal to

$$WW(H_{2m,2q}) = \frac{3}{2} (2q^2 - q) + \left( 5q^2 - \frac{5}{2} q - \frac{13}{6} q m \right) \left( \frac{q}{m} \right)^2 + \left( 2q^2 - q - \frac{7}{2} q m \right) \left( \frac{q}{m} \right)^3 - \frac{2}{3} m q \left( \frac{q}{m} \right)^4$$

$$WW(H_{2m,2q+1}) = (2q + 1) \left[ \frac{2}{3} m \left( \frac{q}{m} \right)^3 + \left( q - \frac{7}{4} q m \right) \left( \frac{q}{m} \right)^2 + \left( \frac{5}{2} q - \frac{13}{12} q m \right) \left( \frac{q}{m} \right)^2 + \frac{3}{2} q \right]$$

**Proof.** Consider the Harary graph $H_{2m,n}$ and refer to Equations 1.3 and 1.4. Thus

$$WW(H_{2m,n})=\frac{1}{2}WW(H_{2m,n})+WW^*(H_{2m,n})$$

where $WW^*(H_{2m,n}) = \sum_{(u,v) \in V(H_{2m,n})} d^2 (u,v)$

Now, suppose $n$ be even ($n=2q$), therefore

$$WW^*(H_{2m,2q}) = 2qm \sum_{d=1}^{\lfloor \sqrt{2q} \rfloor} d^2 + \left( 2q^2 - q - 2qm \left( \frac{q}{m} \right) \left( \frac{q}{m} + 1 \right) \right)^2$$
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\[
W\left(H_{2m,2q}\right) = \frac{2}{6}m \left[ \frac{q}{m} \right] + \frac{5}{2} \left[ \frac{q}{m} \right] + \frac{1}{2} \left[ \frac{q}{m} \right] + \frac{2}{3} \left[ \frac{q}{m} \right] + \frac{7}{2} \left[ \frac{q}{m} \right] + \frac{5}{12} \left[ \frac{q}{m} \right] + \frac{3}{2} \left[ \frac{q}{m} \right] + 1
\]

Thus, \( W\left(H_{2m,2q}\right) = \frac{3}{2} \left( 2q^2 - q \right) + \left( 5q^2 - \frac{5}{2}q - \frac{13}{6}qm \right) \left[ \frac{q}{m} \right] + \left( 2q^2 - q - \frac{7}{2}qm \right) \left[ \frac{q}{m} \right] + \frac{2}{3}mq \left[ \frac{q}{m} \right] \)

If \( n \) be odd (\( n=2q+1 \)), therefore

\[
W\left(H_{2m,2q+1}\right) = m \left( 2q + 1 \right) \sum_{d=1}^{\left\lfloor \frac{\sqrt{n}}{2} \right\rfloor} d^2 + \left( 2q + 1 \right) \left( q - m \left[ \frac{q}{m} \right] \right) \left( \left[ \frac{q}{m} \right] + 1 \right)
\]

Thus, \( W\left(H_{2m,2q+1}\right) = \left( 2q + 1 \right) \left[ \frac{2}{3}m \left[ \frac{q}{m} \right] + \left( q - \frac{3}{2}m \right) \left[ \frac{q}{m} \right] + 2 \right] \left[ \frac{q}{m} \right] + q \)

And theses complete the proof of Theorem 2.4.

**REFERENCES**