HOSOYA POLYNOMIAL, WIENER AND HYPER-WIENER INDICES OF SOME REGULAR GRAPHS

Mohammad Reza Farahani

Department of Applied Mathematics, Iran University of Science and Technology (IUST)
Narmak, Tehran, Iran

ABSTRACT

Let G be a graph. The distance d(u,v) between two vertices u and v of G is equal to the length of a shortest path that connects u and v. The Wiener index W(G) is the sum of all distances between vertices of G, whereas the hyper-Wiener index WW(G) is defined as $WW(G)=\sum_{\{u,v\}\in V(G)}\left(d(v,u)+d(v,u)^2\right)$. *Also, the Hosoya polynomial was introduced by H. Hosoya and define* $H(G,x)=\sum_{\{u,v\}\in V(G)}x^{d(v,u)}$. *In this paper, the Hosoya polynomial, Wiener index and Hyper-Wiener index of some regular graphs are determined.*

KEYWORDS

Network Protocols, Topological distance, Hosoya polynomial, Wiener Index, Hyper-Wiener index, Regular graph, Harary graph.

1. INTRODUCTION

Let $G=(V;E)$ be a simple connected graph. The sets of vertices and edges of G are denoted by $V=V(G)$ and $E=E(G)$, respectively. The distance between vertices u and v of G, denoted by $d(u,v)$, is the number of edges in a shortest path connecting them. An edge $e=uv$ of graph G is joined between two vertices u and v ($d(u,v)=1$). The number of vertex pairs at unit distance equals the number of edges. Also, the topological diameter $D(G)$ is the longest topological distance in a graph G.

A topological index of a graph is a number related to that graph which is invariant under graph automorphism. The Wiener index $W(G)$ is the oldest topological indices, (based structure descriptors) [8], which have many applications and mathematical properties and defined by *Harold Wiener* in 1947 as:

$$W(G)=\frac{1}{2}\sum_{v\in V(G)}\sum_{u\in V(G)}d(v,u) \tag{1.1}$$

Also, for this topological index, there is Hosoya Polynomial. The Hosoya polynomial was introduced by *H. Hosoya*, in 1988 [3] and define as follow:

$$H(G,x)=\frac{1}{2}\sum_{v\in V(G)}\sum_{u\in V(G)}x^{d(v,u)} \tag{1.2}$$

The Hosoya polynomial and Wiener index of some graphs computed [1, 2-4, 5]. Another topological index of graph (based structure descriptor) that was conceived somewhat later is the hyper-Wiener index that introduced by *Milan Randić* in 1993 [6] as

$$WW(G)=\frac{1}{2}\sum_{v\in V(G)}\sum_{u\in V(G)}\left(d(v,u)+d(v,u)^2\right) \tag{1.3}$$

$$=\frac{1}{2}W(G)+\frac{1}{2}\sum_{v\in V(G)}\sum_{u\in V(G)}d(v,u)^2 \tag{1.4}$$

In this paper, we obtained a closed formula of the Hosoya polynomial, Wiener and Hyper-Wiener indices for an interesting regular graph that called *Harary graph*. The general form of the Harary graph $H_{2m,n}$ is defined as follows:

Definition 1.1 [7]. Let m and n be two positive integer numbers, then the Harary graph $H_{2m,n}$ is constructed as follows:

It has vertices $1,2,\ldots,n-1,n$ and two vertices i and j are joined if $i-m \le j \le i+m$ (where addition is taken modulo n).

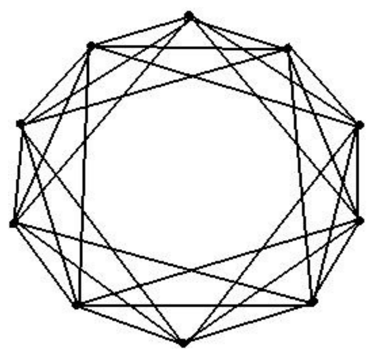

Figure1. The Harary graph $H_{6,10}$.

2. MAIN RESULTS

In this section we compute the Hosoya polynomial, Wiener index and hyper-Wiener index of the Harary graph $H_{2m,n}$.

Theorem 2.1. Consider the Harary graph $H_{2m,n}$ for all positive integer number m and n. Then,

- The Hosoya polynomial of $H_{2m,n}$ for n odd is

$$H_{2m,n}=\sum_{d=1}^{\left[n/2m\right]} mnx^d + n\left(\left[n/2\right]-m\times\left[n/2m\right]\right)x^{\left[n/2m\right]+1}$$

- The Hosoya polynomial of $H_{2m,n}$ for n even ($=2q$) is

$$H_{2m,2q}=\sum_{d=1}^{\left[q/m\right]} 2qmx^d + 2q^2 - 2qm\left[q/m\right] - q\,x^{\left[q/m\right]+1}$$

Proof. Let H be the Harary graph $H_{2m,n}$ (Figure 1). The number of vertices in this graph is equal to $|V(H_{2m,n})|=n$ ($\forall m,n\in$□). Also, two vertices $v_i,v_j\in V(H_{2m,n})$ are adjacent if and only if $|i-j|\leq m$, then $d(v_i,v_j)=1$ thus the number of edges of this regular graph is $|E(H_{2m,n})|=\frac{n(2m)}{2}=mn$.

It is obvious that for all graph G, $H(G,1)=\sum_{i=1}^{d(G)} d(G,i)=\binom{n}{2}=\frac{n(n-1)}{2}$. On the other hand from the structure of $H_{2m,n}$ (Figure 1), we see that all vertices of $H_{2m,n}$ have similar geometrical and topological conditions then the number of path as distance i in $H_{2m,n}$ ($d(H_{2m,n},i)$) is a multiple of the number of vertices (n). In other words, any vertex $v\in V(H_{2m,n})$ is an endpoint of $d(H_{2m,n},i)/n$ paths as distance i in $H_{2m,n}$ ($=d_v(H_{2m,n},i)$). For example $\forall v\in V(H_{2m,n})$, there are $2m=d_v(H_{2m,n},1)$ paths as distance 1 (or all edges incident to v) in $H_{2m,n}$. From Figure 1, one can see that $\forall i=1,2,\ldots,n$ and $v_i,v_{i+m}\in V(H_{2m,n})$, $d(v_i,v_{i\pm m+j})=2$ ($j=\{1,2,\ldots,m\}$) because $v_iv_{i\pm m}$, $v_{i\pm m}v_{i\pm m+1}\in E(H_{2m,n})$ and $d(v_i,v_{i\pm m})=d(v_{i\pm m},v_{i\pm m+1})=1$. Obviously $d_{vi}(H_{2m,n},2)=2m$

Now, since $\forall i\in\{1,2,\ldots,n\}$ and $v_i,v_{i+m}\in V(H_{2m,n})$, $d(v_i,v_{i\pm m})=d(v_{i\pm m},v_{i\pm 2m})=d(v_{i\pm 2m},v_{i\pm 3m})=\ldots=d(v_{i\pm(k-1)m},v_{i\pm km})=1$ such that $k\leq[n/2m]$. Thus $\forall d=1,\ldots,k$ $d(v_i,v_{i\pm dm})=d$ and obviously $d(v_i,v_{i\pm dm+j})=d+1$ ($j=1,2,\ldots,m-1$). By these mentions, one can see that for an arbitrary vertex v_i the diameter $D(H_{2m,n})$ of this Harary graph is $d(v_i,v_{i+[n/2]})=[n/2m]+1$. Also the distance between vertices v_i and $v_{i\pm([n/2r]m+j)}$ is equal to $D(H_{2m,n})$ for all $j=1,2,\ldots,[n/2]$. It's easy to see that if n be odd then $\forall v_i\in V(H_{2m,n})$, $d_{vi}(H,[n/2m]+1)=|[n/2]-m\times[n/2m]|$, else n be even then $d(H,d(H))=\frac{n}{2}(|\left[\frac{n}{2}\right]-m\left[\frac{n}{2m}\right]|-1)$.

Now by using the definition of Hosoya polynomial (equation 2), we have

$$\begin{aligned}
H(H_{2m,n},x) &= \sum_{\{u,v\}\in V(H_{2m,n})} x^{d(v,u)} \\
&= \sum_{i,j=1}^{n} x^{d(v_i,v_j)} \\
&= \sum_{i=1}^{n}\sum_{j=0}^{m-1}\sum_{k=0}^{\left[\frac{n}{2m}\right]} x^{d(v_i,v_{i\pm k+j})} \\
&= mnx^1+mnx^2+\ldots+mnx^{\left[n/2m\right]}+d(H_{2m,n},\left[n/2m\right]+1)x^{\left[n/2m\right]+1}
\end{aligned}$$

where $d(H_{2m,n},\left[n/2m\right]+1)=\begin{cases} n\times\left|m\times\left[\frac{n}{2m}\right]-\left[\frac{n}{2}\right]\right| & n \quad odd \\ n\times\left|m\times\left[\frac{n}{2m}\right]-\left[\frac{n}{2}\right]\right|-\frac{n}{2} & n \quad even \end{cases}$

Here the proof of Theorem 2.1 is completed.

Theorem 2.2. Let $H_{2m,n}$ be the Harary graph. Then the Wiener index of $H_{2m,n}$ is equal to

- If n be odd, $W(H_{2m,2q+1})=(2q+1)\left(q+\left[q/m\right]\left(q-\frac{m}{2}\right)-\frac{m}{2}\left[q/m\right]^2\right)$
- If n be even, $W(H_{2m,2q})=(2q^2-q)+q\left[q/m\right](2q-1-m)-qm\left[q/m\right]^2$

Proof. By according to the definitions of the Wiener index and Hosoya Polynomial of a graph G (Equation 1.1 and 1.2), then the Wiener index $W(G)$ will be the first derivative of Hosoya polynomial $H(G,x)$ evaluated at $x=1$. Thus by using the results from Theorem 2.1, we have

1- If n be an arbitrary even positive integer number ($n=2q$), then

$$W(H_{2m,2q})=\sum_{\{u,v\}\in V(H_{2m,2q})} d(v,u)$$
$$=\left.\frac{\partial H(H_{2m,2q},\mathrm{x})}{\partial x}\right|_{x=1}$$
$$=\frac{\partial\sum_{d=1}^{[q/m]}2qmx^{d}+2q^{2}-2qm\left[q/m\right]-q.x^{[q/m]+1}}{\partial x}\Big|_{x=1}$$
$$=2qm\sum_{d=1}^{[q/m]}d+\left(-2qm\left[q/m\right]^{2}+\left[q/m\right]\left(2q^{2}-q-2qm\right)+\left(2q^{2}-q\right)\right)$$
$$=\left(2q^{2}-q\right)+q\left(2q-1-m\right)\left[q/m\right]-qm\left[q/m\right]^{2}$$

2- If n be an odd positive integer number ($n=2q+1$), then

$$W(H_{2m,2q+1})=\frac{\partial\sum_{d=1}^{[n/2m]}mnx^{d}+n\left(\left[n/2\right]-m\times\left[n/2m\right]\right)x^{[n/2m]+1}}{\partial x}\Big|_{x=1}$$
$$=\left(2q+1\right)\left(\sum_{d=1}^{[n/2m]}m\times d+\left(q-m\left[q/m\right]\right)\left(\left[q/m\right]+1\right)\right)$$
$$=\left(2q^{2}+q\right)+\left[q/m\right]\left(2q^{2}+q-\frac{m\left(2q+1\right)}{2}\right)-\frac{m\left(2q+1\right)}{2}\left[q/m\right]^{2}.$$

Corollary 2.3. Suppose n be an even positive integer number ($n=2q$) and $2m|n$, then such that $q/m=r$. Thus $W(H_{2m,2mr})=mr(mr^{2}+mr-r-1)$.

Theorem 2.4. The hyper-Wiener index of the Harary graphs $H_{2m,2q}$ and $H_{2m,2q+1}$ are equal to

$$WW(H_{2m,2q})=\frac{3}{2}\left(2q^{2}-q\right)+\left(5q^{2}-\frac{5}{2}q-\frac{13}{6}qm\right)\left[q/m\right]+\left(2q^{2}-q-\frac{7}{2}qm\right)\left[q/m\right]^{2}-\frac{2}{3}mq\left[q/m\right]^{3}$$
$$WW(H_{2m,2q+1})=\left(2q+1\right)\left[-\frac{2}{3}m\left[q/m\right]^{3}+\left(q-\frac{7}{4}m\right)\left[q/m\right]^{2}+\left(\frac{5}{2}q-\frac{13}{12}m\right)\left[q/m\right]+\frac{3}{2}q\right]$$

Proof. Consider the Harary graph $H_{2m,n}$ and refer to Equations 1.3 and 1.4. Thus

$WW(H_{2m,n})=\frac{1}{2}W(H_{2m,n})+WW^{*}(H_{2m,n})$

where $WW^{*}(H_{2m,n})=\sum_{\{u,v\}\in V(H_{2m,n})}d^{2}(v,u)$

Now, suppose n be even ($n=2q$), therefore

$$WW^{*}(H_{2m,2q})=2qm\sum_{d=1}^{[q/m]}d^{2}+\left(2q^{2}-q-2qm\left[q/m\right]\right)\left(\left[q/m\right]+1\right)^{2}$$

$$=\frac{2qm}{6}\left(2\left[q/m\right]^3+3\left[q/m\right]^2+\left[q/m\right]\right)+\left(2q^2-q-2qm\left[q/m\right]\right)\left(\left[q/m\right]^2+2\left[q/m\right]+1\right)$$

$$=\left(2q^2-q\right)+\left(4q^2-2q-\frac{5}{3}qm\right)\left[q/m\right]+\left(2q^2-q-3qm\right)\left[q/m\right]^2-\frac{2}{3}mq\left[q/m\right]^3$$

Thus, $WW(H_{2m,2q})=\frac{3}{2}\left(2q^2-q\right)+\left(5q^2-\frac{5}{2}q-\frac{13}{6}qm\right)\left[q/m\right]+\left(2q^2-q-\frac{7}{2}qm\right)\left[q/m\right]^2-\frac{2}{3}mq\left[q/m\right]^3$

If n be odd ($n=2q+1$), therefore

$$WW^*(H_{2m,2q+1})=m\left(2q+1\right)\sum_{d=1}^{[q/m]}d^2+\left(2q+1\right)\left(q-m\left[q/m\right]\right)\left(\left[q/m\right]+1\right)^2$$

$$=\frac{(2q+1)m}{6}\left(2\left[q/m\right]^3+3\left[q/m\right]^2+\left[q/m\right]\right)+(2q+1)\left(q-m\left[q/m\right]\right)\left(\left[q/m\right]^2+2\left[q/m\right]+1\right)$$

$$=\left(2q+1\right)\left[-\frac{2}{3}m\left[q/m\right]^3+\left(q-\frac{3}{2}m\right)\left[q/m\right]^2+\left(2q-\frac{5}{6}m\right)\left[q/m\right]+q\right]$$

Thus, $WW(H_{2m,2q+1})=\left(2q+1\right)\left[-\frac{2}{3}m\left[q/m\right]^3+\left(q-\frac{7}{4}m\right)\left[q/m\right]^2+\left(\frac{5}{2}q-\frac{13}{12}m\right)\left[q/m\right]+\frac{3}{2}q\right].$

And theses complete the proof of Theorem 2.4.

REFERENCES

[1] M.V. Diudea. Hosoya polynomial in Tori. MATCH Commun. Math. Comput. Chem. 45, (2002), 109-122.

[2] A.A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: Theory and applications, Acta Appl. Math. 66 (2001), 211–249.

[3] H. Hosoya. On some counting polynomials in chemistry. Discrete Appl. Math. 19, (1989), 239-257.

[4] D.J. Klein, I. Lukovits, I. Gutman, On the definition of the hyper-Wiener index for cycle-containing structures, J. Chem. Inf. Comput. Sci. 35 (1995), 50–52.

[5] M. Knor, P. Potočnik and R. Škrekovski. Wiener index of iterated line graphs of trees homeomorphic to the claw K1;3. Ars Math. Contemp. 6 (2013), 211–219

[6] M. Randić, Novel molecular descriptor for structure-property studies. Chem. Phys. Lett., 211, (1993),478.

[7] B. West, Introduction to graph theory. Prentice Hall of India, (2003).

[8] H. Wiener, Structural determination of paraffin boiling points, J. Amer. Chem. Soc. 69 (1947), 17–20.

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