SEASONAL VARIATION OF CARRYING CAPACITY ON DISEASED MODEL CAN CAUSE SPECIES EXTINCTION

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ABSTRACT

Most natural populations experience fluctuations in biological and environmental factors which causes carrying capacity variation. In this paper, we have introduced a diseased prey-predator model with periodically varying carrying capacity. We have studied the effects of different amplitudes of oscillaton as well as different frequencies of oscillation on the dynamics of the model. We have done bifurcation analysis of the model with respect to the amplitude of oscillation and frequency of oscillation of the carrying capacity. We observe limit cycle, low periodic orbits, high periodic orbits and chaos in the model. We observe the existence of critical frequency and amplitude of oscillation of carrying capacity for which the prey population extinct. Through bifurcation analysis we observe oscillatory coexistence of species in the model. Our results confirm that amplitude and frequency of oscillation of carrying capacity are key parameters together with the force of infection and body size of intermediate predator in a diseased prey-predator model.

KEYWORDS

Limit cycle, Period-2 orbit, Period-3 orbit, Chaos, Birfurcation.

1. INTRODUCTION

One of the most exciting modern applications of mathematics are modelling and analysis of biological and ecological systems. There are many two dimensional models eg. Lotka [1] and Volterra [2] predator- prey model, Rosenzweig-MacArthur [3] model, Murray [4] model etc. But a two dimensional model is very poor to capture the dynamical complexity of real food chain. That is why, three dimensional model is more appropriate to study the food chain system. There is a large literature on these models. Researchers have modified the food chain model introducing various factors and various types of functional responses. Hastings and Powell [5] introduce a continuous time model of a food chain incorporating nonlinear functional responses. Hsu et al. [6] analyze a tritrophic ratio dependent food chain model and discuss the extinction dynamics of the model. Maiti et al. [7] study a tritrophic food chain model with mixed selection of functional responses. Lobry and Mazenc [8] discuss the dynamical nature of a chemostat model with an arbitrary number of species. Wen [9] examine the sufficient conditions for existence of periodic solutions of a food chain model with Holling type-II functional response on time scales. Mandal et al. [10] analytically investigate the local qualitative temporal behaviour around inner equilibrium point of a three species model. Kooi et al. [11] study the ability of top predator to invade the food chain in a chemostat. Hsu and Waltman [12] discuss the dynamics of a chemostat model with an external nutrient and with an external inhibitor using theory of monotone flows.

Again harvesting technics, omnivory criteria, toxin factors etc. are used in the progress of ecological system. Mukhopadhyay [13] study the autotroph-herbivore food chain model with Holling type-II harvest function in the presence of decomposers. Kar and Matsuda [14] analyze the effect of harvesting efforts on the prey-predator system with Holling type-III functional response. Tanabe and Namba [15] discuss the chaotic nature of a tritrophic food web model with omnivory and show that intraguild predation can destabilize food webs and induce chaos. Diehl and Feibel [16] reported that how enrichment affect coexistence of the species on three level food chain model with omnivory. Upadhyay [17] investigate dynamic nature of a three dimensional food chain model with Holling type-II functional response of toxin liberation process and suggest that toxic substences may act as bio-control by changing the state of chaos to order. By including the effect of toxin Zhu et al. [18] prove the existence of limit cycles in the 3-D system. S. Vaidyanathan [22] describes chaotic nature of a system using hybrid synchronization. Ghosh et al. [23] dicuss the reliability of a system through Monte-Carlo simulation. Again Mishra and Mankar [24] discuss a model using chaotic map.

Introducing disease in ecological system, a new branch eco-epidemiology studies are in progress. Mukhopadhyay and Bhattacharyya [19] investigate the role of predator switching on the dynamics of a diseased eco-epidemiological model. Arino et al. [20] prove that introduction of an infected population in the classical ratio-dependent predator-prey model may act as a biological control to save the population from extinction. Das et al. [21] modified Hastings and Powell's[5] model by introducing disease in the prey population. They show that disease in prey population and body size of intermediate predator can control the chaotic dynamics. They have assumed carrying capacity of the model as constant.

Fluctuations in biological and environmental factors in natural populations causes carrying capacity variation. As far as our knowledge goes none of the studies was done on diseased prey population with periodically varying carrying capacity. But variation in carrying capacity is important in managing harvest of species and for planning carrying capacity research. In this paper, we investigate the effects of periodically varying carrying capacity on diseased prey population model. We have done bifurcation analysis of the model with respect to the amplitude of oscillation and frequency of oscillation of the carrying capacity. We have discussed the effects of different amplitudes of oscillatons and different frequencies of oscillation of carrying capacity on the dynamics of the food chain.

This paper has been organized as follows. In Section 2, we dicuss our model, in Section 3, we discuss the simulation results and in Section 4 we concluded the main results of our model.

2. MODEL

Recently, Das et al.[21] proposed predator-prey model with disease in prey population as

$$\begin{aligned}
\frac{dS}{dT} &= RS\left(1 - \frac{S}{K}\right) - \alpha IS - C_1A_1\frac{P_1S}{(B_1+S)} \\
\frac{dI}{dT} &= \alpha IS - A_2P_1I - D_1I \\
\frac{dP_1}{dT} &= A_1\frac{P_1S}{(B_1+S)} + C_2A_2P_1I - A_3\frac{P_1P_2}{(B_2+P_1)} - D_2P_1 \\
\frac{dP_2}{dT} &= C_3A_3\frac{P_1P_2}{(B_2+P_1)} - D_3P_2
\end{aligned} \qquad (1)$$

Here S, I, P_1, P_2 are respectively the susceptible prey population, infected prey population, the intermediate predator population, top predator population. A_1 and A_2 are the maximal predation rate of intermediate predator for susceptible and infected prey respectively; A_3 is the maximal predation rate of top predator for intermediate predator; B_1 and B_2 are the half saturation constant for functional response of intermediate and top predator respectively; C_1^{-1} is the conversion rate of susceptible prey to intermediate predator; C_2 is the conversion rate of infected prey to intermediate predator; C_3 is the conversion rate of intermediate predator to top predator. D_1, D_2 and D_3 are respectively the rates of death of infected prey population, intermediate predator and top predator, K is the carrying capacity of the system.

Introducing the dimensionless parameters as $s = \frac{S}{K}$, $i = \frac{I}{K}$, $p_1 = \frac{P_1}{K}$, $p_2 = \frac{P_2}{K}$ and t = TR, the model (1) becomes:

$$\begin{aligned}
\frac{ds}{dt} &= s(1- s) - \alpha K\frac{is}{R} - \frac{C_1A_1}{R}\frac{Kp_1s}{(B_1+Ks)} \\
\frac{di}{dt} &= \alpha K\frac{is}{R} - \frac{A_2K}{R}ip_1 - \frac{D_1i}{R} \\
\frac{dp_1}{dt} &= \frac{A_1}{R}\frac{Kp_1s}{(B_1+Ks)} + C_2\frac{A_2K}{R}ip_1 - \frac{A_3K}{R}\frac{p_1p_2}{(B_2+p_1K)} - \frac{D_2p_1}{R} \\
\frac{dp_2}{dt} &= C_3\frac{A_3K}{R}\frac{p_1p_2}{(B_2+p_1K)} - \frac{D_3p_2}{R}
\end{aligned} \qquad (2)$$

Since carrying capacity of an ecological system is season dependent, we assume that the carrying capacity of the system vary sinusiodally with time. Mathematically this means carrying cpacity K of the system will be replaced by $K = K_0(1 + q\sin(wt))$. Here q is the amplitude of oscillation, w is the frequency of oscillation and K_0 is a dimensionless number, q can take any value between 0 to 1. With this modification we obtained the following model:

$$\frac{ds}{dt} = s(1 - s) - a(1 + q\sin(p_3))si - b(1 + q\sin(p_3))\frac{p_1s}{1 + cs(1 + q\sin(p)_3)}$$

$$\frac{di}{dt} = a(1 + q\sin(p_3))si - d(1 + q\sin(p_3))p_1i - ei$$

(3)

$$\frac{dp_1}{dt} = f(1 + q\sin(p_3))\frac{sp_1}{(1 + cs(1 + q\sin(p_3)))} + g(1 + q\sin(p_3))p_1 i - h(1 + q\sin(p_3))$$

$$\frac{p_2 p_1}{(1 + mp_1(1 + q\sin(p_3)))} - jp_1$$

$$\frac{dp_2}{dt} = k(1 + q\sin(p_3))\frac{p_2 p_1}{(1 + mp_1(1 + q\sin(p_3)))} - lp_2$$

$$\frac{dp_3}{dt} = w$$

Where $a = \frac{\alpha K_0}{R}$, $b = \frac{A_1 C_1 K_0}{RB_1}$, $c = \frac{K_0}{B_1}$, $d = \frac{K_0 A_2}{R}$, $e = \frac{D_1}{R}$, $f = \frac{A_1 K_0}{RB_1}$, $g = \frac{A_2 C_2 K_0}{R}$, $h = \frac{A_3 K_0}{RB_2}$, $m = \frac{K_0}{B_2}$, $j = \frac{D_2}{R}$, $l = \frac{D_3}{R}$, $k = \frac{A_3 C_3 K_0}{RB_2}$, $p_3 = wt$.

3. SIMULATION RESULTS

We have solved numerically the system (3) using fourth order Runge-Kutta method with a hypothetical set of parameter values most of which are taken from Das et al. [21] model. The following parameter values are kept fixed throughout the numerical simulations, we have choosen a = 1.3, b = 5.0, c = 2.3, d = 3.0, e = 0.5, f = 5.0, g = 2.5, h = 0.1, m = 2.0, j = 0.4, k = 0.1, l = 0.01.

We draw the phase diagram of total prey population vs. intermediate predator for different values of amplitude of oscillation q of carrying capacity in Figure-1 and Figure-2 keeping frequency of oscillation w = 0.1 fixed. From the figures we observe that for constant carrying capacity as well as for small amplitude oscillating of carrying capacity the total prey population vs. intermediate predator population has period-2 oscillation. But for higher amplitude of oscillation of carrying capacity the phase diagram of total prey population vs. intermediate predator population shows period-4 orbit, high periodic orbit, chaotic orbit, period-3 and period-6 orbits etc. But interestingly enough for q = 1 this phase diagram shows period-2 orbit again. We have also investigated the effects of variation of frequency of oscillation w of carrying capacity keeping the amplituide of oscillation q = 0.25 fixed in Figure-3. From the figure it is observed that for frequency of oscillation w = 0.4 we obtain limit cycle, for w = 0.6 we obtain chaotic behaviour, for w = 0.8 again limit cycle and for w = 1.0 again chaotic behaviour in the phase space of total prey vs. intermediate predator.

We have done bifurcation analysis of the system with respect to the bifurcation parameter q, keeping frequency of oscillation fixed at w = 0.1 taking different values of force of infection a. From the bifurcation diagram (Figure-4) we observe that total prey population varries significantly with respect to the amplitude of oscillation of the carrying capacity. Taking the force of infaction a = 1.3, a = 2.0 and a = 3.2 we observe that there exist some critical frequncy and critical amplitude of oscillation of carrying capacity for which prey species extinct as seen from Figure-4 to Figure-6. We have also done bifurcation analysis of the system with respect to the frequency of oscillation w as bifurcation parameter keeping the amplitude of oscillation fixed at q = 0.25 in Figure-7 for a = 1.3, in Figure-8 for a = 2.0 and in Figure-9 for a = 3.2. We observe limit cycle, period-2, period-6 and chaotic orbit of the system.

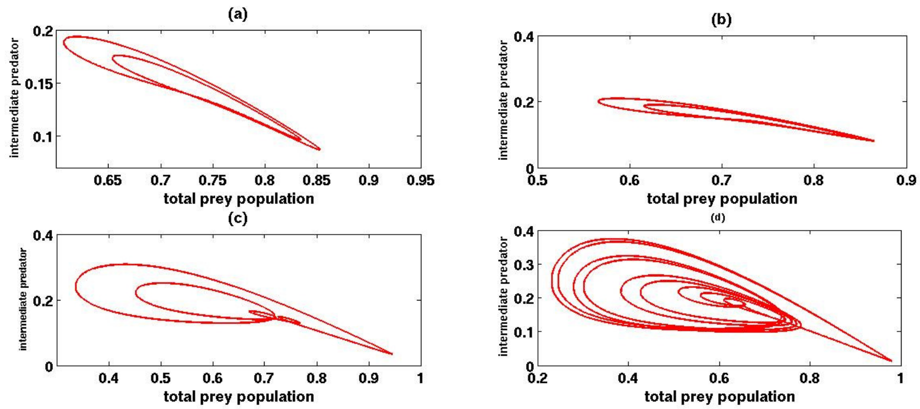

Figure 1: Phase diagram (a) for q = 0 and w = 0, (b) for q = 0.01 and w = 0.1, (c) for q = 0.05 and w = 0.1 and (d) for q = 0.1 and w = 0.1.

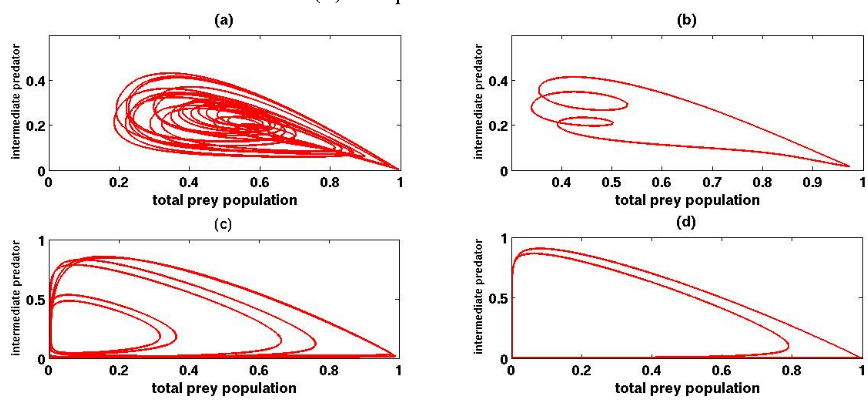

Figure 2: Phase diagram (a) for q = 0.25 and w = 0.1, (b) for q = 0.5 and w = 0.1, (c) for q = 0.86 and w = 0.1 and (d) for q = 1.0 and w = 0.1.

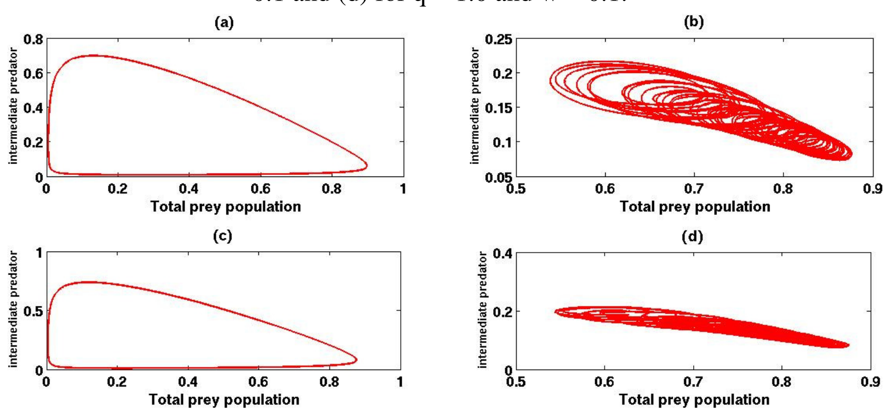

Figure 3: Phase diagram (a) for q = 0.25 and w = 0.4, (b) for q = 0.25 and w = 0.6, (c) for q = 0.25 and w = 0.8 and (d) for q = 0.25 and w = 1.0.

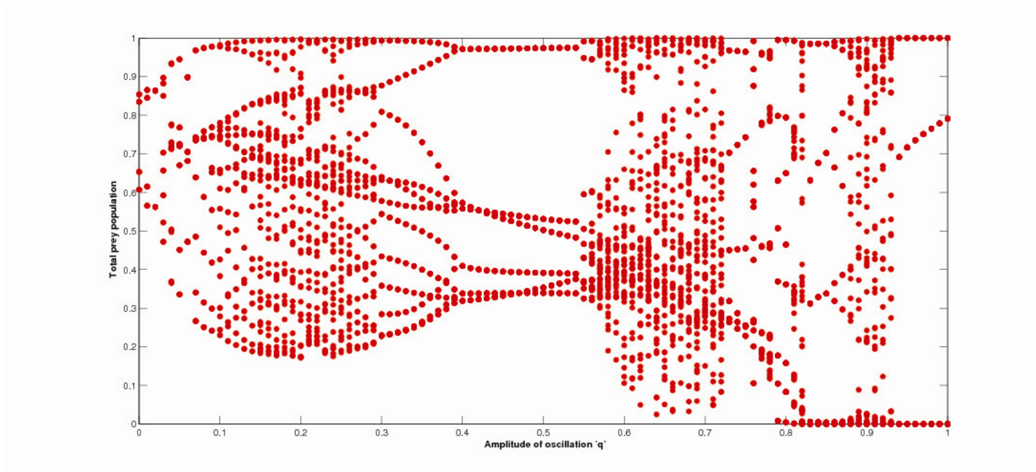

Figure 4: Bifurcation diagram of total prey population with respect to amplitude of oscilation ‘q’ varying from 0 to 1.

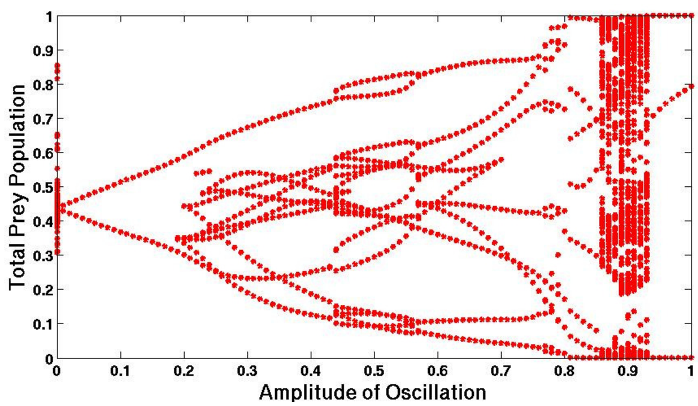

Figure 5: Bifurcation diagram of total prey population with respect to amplitude of oscillation ‘q’ varying from 0 to 1 taking a = 2.0 and w = 0.1.

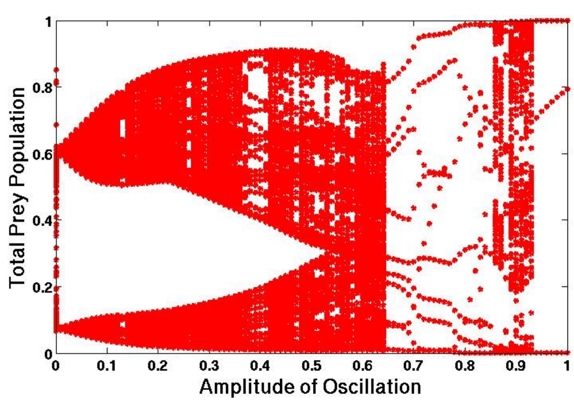

Figure 6: Bifurcation diagram of total prey population with respect to amplitude of oscillation ‘q’ varying from 0 to 1 taking a = 3.2 and w = 0.1.

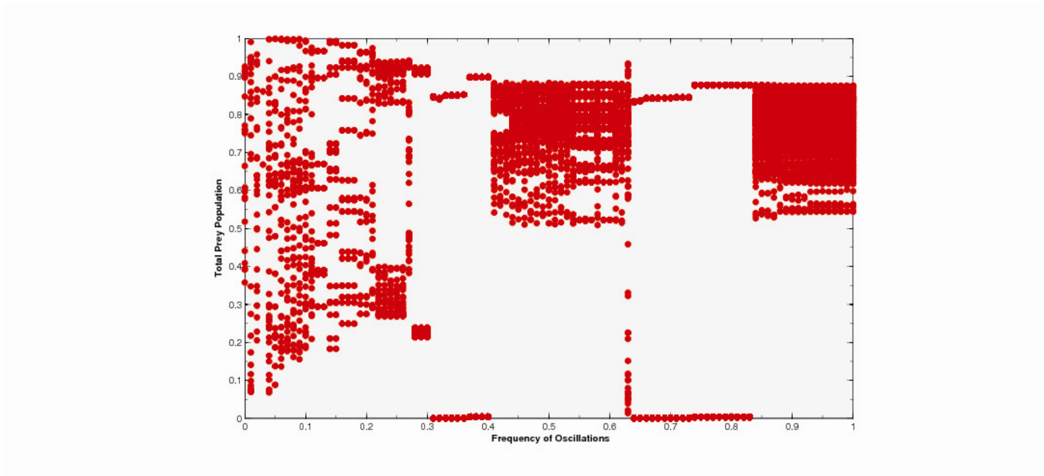

Figure 7: Bifurcation diagram of total prey population with respect to frequency of oscillation ‘w’ varying from 0 to 1 taking a = 1.3 and q = 0.25

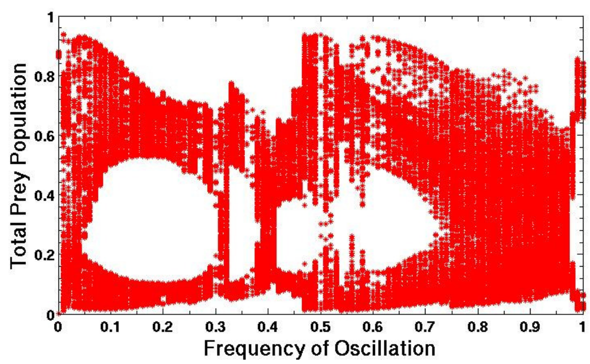

Figure 8: Bifurcation diagram of total prey population with respect to frequency of oscillation ‘w’ varying from 0 to 1 taking a = 2.0 and q = 0.25.

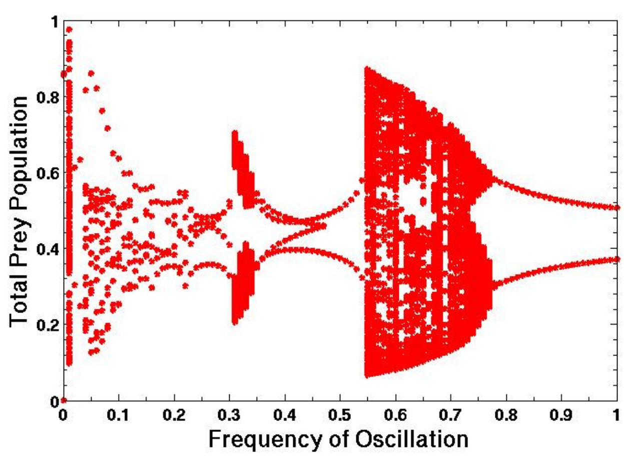

Figure 9: Bifurcation diagram of total prey population with respect to frequency of oscillation w varying from 0 to 1 taking a = 3.2 and q = 0.25.

4. CONCLUTION

We have introduced a diseased prey-predator model with periodically varying carrying capacity. We have studied the effects of different amplitudes of oscillatons and different frequencies of oscillation on the dynamics of the diseased food chain. We observe that the dynamics of the model is highly sensitive to variation of carrying capacity. Therefore improved food chain model must have the carrying capacity variation to capture the actual dynamics of real food chain. Under considering seasonal variation of carrying capacity we obtained oscillatory coexistence of predator-prey system. The variation of amplitude and frequency of oscillation of carrying capacity in the model is sufficient to obtain period-2, period-4, period-6 oscillation and chaos keeping the force of infection fixed. From the bifurcation diagrams we observe that there exist some critical frequency of oscillation as well as critical amplitude of oscillation of carrying capacity for which the prey population is going to extinct. Therefore it is observed that oscillation in the carrying capacity can cause species extinction. With constant carrying capacity Das et.al.[21] predicted stable coexistence for force of infection a in $2.0 \le a \le 3.2$ but with periodically varrying carrying capacity we have shown the possibility of species extinction there. There exist some frequency and amplitude of oscillation of carrying capacity for which stable oscillatory coexitence is also possible in $2.0 \le a \le 3.2$. Therefore the dynamics of our model is qualitatively distinct from the model [21]. Our results demonstrate that not only disease in prey population and body size of intermediate predator are the key parameters for controlling the chaotic dynamics but also the amplitude and frequency of oscillation of carrying capacity play an important role diseased prey-predator model. Temporal variation in carrying capacity is important in managing harvesting of species and for planning carrying capacity research. Carrying capacity as a function of not only time but also many ecological parameters must be understood in future.

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