ADAPTIVE CONTROL AND SYNCHRONIZATION OF THE CAI SYSTEM

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ABSTRACT

This paper derives new results for the adaptive control and synchronization design of the Cai system (2007), when the system parameters are unknown. Cai system is one of the paradigms of 3-D chaotic systems discovered by Cai and Tan (2007). In this paper, we first construct an adaptive controller to stabilize the Cai system to its unstable equilibrium at the origin. Then we build an adaptive synchronizer to achieve global chaos synchronization of the identical Cai systems with unknown parameters. The results derived for adaptive stabilization and adaptive synchronization for the Cai systems have been established using adaptive control theory and Lyapunov stability theory. Numerical simulations have been shown to demonstrate the effectiveness of the adaptive control and synchronization schemes derived in this paper for the Cai system.

KEYWORDS

Adaptive Control, Chaos, Chaotic Systems, Synchronization, Cai System.

1. INTRODUCTION

Nonlinear dynamical systems, which are extremely sensitive to changes in initial conditions, are known as chaotic systems. Chaotic systems exhibit random-like behaviour in its deterministic motion. Experimentally, chaos was first discovered by Lorenz ([1], 1963), while he was simulating weather models. A chaotic system simpler than the Lorenz system was proposed by Rössler ([2], 1976).

The control of chaotic systems is to design state feedback control laws that stabilizes the chaotic systems around the unstable equilibrium points. Active control technique is used when the system parameters are known and adaptive control technique is used when the system parameters are unknown [3-4].

Synchronization of chaotic systems is a phenomenon that may occur when two or more chaotic attractors are coupled or when a hyperchaotic attractor drives another hyperchaotic attractor. In the last two decades, there has been significant interest in the literature on the synchronization of chaotic and hyperchaotic systems [5-16].

In 1990, Pecora and Carroll [5] introduced a method to synchronize two identical chaotic systems and showed that it was possible for some chaotic systems to be completely synchronized. From then on, chaos synchronization has been widely explored in a variety of fields including physical systems [6], chemical systems [7], ecological systems [8], secure communications [9-10], etc.

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The pioneering work by Pecora and Carroll (1990) has been followed by a variety of impressive approaches in the literature such as the sampled-data feedback method [11], OGY method [12], time-delay feedback method [13], backstepping method [14], active control method [15-20], adaptive control method [21-25], sliding mode control method [26-28], etc.

This paper is organized as follows. In Section 2, we give a description of the Cai chaotic system (Sprott, [29], 1994). In Section 3, we derive results for the adaptive control of Cai chaotic system with unknown parameters. In Section 4, we derive results for the adaptive synchronization of the identical Cai chaotic systems with unknown parameters. Section 5 contains a summary of the main results derived in this paper.

2. SYSTEM DESCRIPTION

The Cai system ([29], 2007) is described by the 3D dynamics

\[\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= bx_1 + cx_2 - x_1 x_3 \\
\dot{x}_3 &= x_1^2 - dx_3
\end{align*}\]

where \(x_1, x_2, x_3\) are the state variables of the system and \(a, b, c, d\) are constant, positive parameters of the system.

The system (1) is chaotic when the parameter values are taken as

\[a = 20, \ b = 14, \ c = 10.6 \ \text{and} \ d = 2.8\]

Figure 1 describes the strange attractor of the Cai system (1).

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Figure 1. The Strange Attractor of the Cai System
When the parameter values are taken as in (2) for the Cai chaotic system (1), the system linearization matrix at the equilibrium point $E_0 = (0, 0, 0)$ is given by

$$A = \begin{bmatrix} -20 & 20 & 0 \\ 14 & 10.6 & 0 \\ 0 & 0 & -2.8 \end{bmatrix}$$

which has the eigenvalues

$$\lambda_1 = -27.3736, \quad \lambda_2 = -2.8 \quad \text{and} \quad \lambda_3 = 17.9736$$

Since $\lambda_3$ is an unstable eigenvalue of $A$, it follows from Lyapunov stability theory [30] that the Cai system (1) is unstable at the equilibrium point $E_0 = (0, 0, 0)$.

### 3. Adaptive Control of the Cai Chaotic System

#### 3.1 Theoretical Results

In this section, we design adaptive control law for globally stabilizing the Cai system (2007), when the parameter values are unknown.

Thus, we consider the controlled Cai system, which is described by the 3D dynamics

$$\dot{x}_1 = a(x_2 - x_1) + u_1$$
$$\dot{x}_2 = bx_1 + cx_2 - x_1x_3 + u_2$$
$$\dot{x}_3 = x_1^2 - dx_3 + u_3$$

where $u_1, u_2$ and $u_3$ are feedback controllers to be designed using the states $x_1, x_2, x_3$ and estimates $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ of the unknown parameters $a, b, c, d$ of the system.

In order to ensure that the controlled system (3) globally converges to the origin asymptotically, we consider the following adaptive control functions

$$u_1 = -\hat{a}(x_2 - x_1) - k_1 x_1$$
$$u_2 = -\hat{b}x_1 - \hat{c}x_2 + x_1x_3 - k_2 x_2$$
$$u_3 = -x_1^2 + \hat{d}x_3 - k_3 x_3$$

where $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are estimates of the parameters $a, b, c$ and $d$, respectively, and $k_i, (i = 1, 2, 3)$ are positive constants.

Substituting the control law (4) into the controlled Cai dynamics (3), we obtain
\[
\dot{x}_1 = (a - \hat{a}) (x_2 - x_i) - k_i x_i \\
\dot{x}_2 = (b - \hat{b}) x_i + (c - \hat{c}) x_2 - k_2 x_2 \\
\dot{x}_3 = -(d - \hat{d}) x_3 - k_3 x_3
\]

Let us now define the parameter errors as

\[
e_a = a - \hat{a}, \quad e_b = b - \hat{b}, \quad e_c = c - \hat{c} \quad \text{and} \quad e_d = d - \hat{d}
\]

Using (6), the closed-loop dynamics (5) can be written compactly as

\[
\dot{x}_1 = e_a (x_2 - x_i) - k_i x_i \\
\dot{x}_2 = e_b x_i + e_c x_2 - k_2 x_2 \\
\dot{x}_3 = -e_d x_3 - k_3 x_3
\]

Next, we consider the quadratic Lyapunov function

\[
V(x, x, x, e_a, e_b, e_c, e_d) = \frac{1}{2} \left( x_i^2 + x_2^2 + x_3^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2 \right)
\]

which is a positive definite function on \( \mathbb{R}^7 \).

Note also that

\[
\dot{e}_a = -\hat{a}, \quad \dot{e}_b = -\hat{b}, \quad \dot{e}_c = -\hat{c}, \quad \dot{e}_d = -\hat{d}
\]

Differentiating \( V \) along the trajectories of (7) and using (9), we obtain

\[
\dot{V} = -k_i x_i^2 - k_2 x_2^2 - k_3 x_3^2 + e_a \left[ x_i (x_2 - x_i) - \hat{a} \right] + e_b \left( x_i x_2 - \hat{b} \right) \\
+ e_c \left( x_2^2 - \hat{c} \right) + e_d \left( -x_3^2 - \hat{d} \right)
\]

In view of Eq. (10), the estimated parameters are updated by the following law:

\[
\dot{\hat{a}} = x_i (x_2 - x_i) + k_4 e_a \\
\dot{\hat{b}} = x_i x_2 + k_5 e_b \\
\dot{\hat{c}} = x_2^2 + k_6 e_c \\
\dot{\hat{d}} = -x_3^2 + k_7 e_d
\]

where \( k_4, k_5, k_6 \) and \( k_7 \) are positive constants.

Next, we prove the following result.
Theorem 1. The controlled Cai system (1) with unknown parameters is globally and exponentially stabilized for all initial conditions \( x(0) \in \mathbb{R}^3 \) by the adaptive control law (4), where the update law for the parameters is given by (11) and \( k_i, \ (i = 1, \ldots, 7) \) are positive constants.

Proof. Substituting (11) into (10), we get

\[
\dot{V} = -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 e_a^2 - k_5 e_b^2 - k_6 e_c^2 - k_7 e_d^2
\]

which is a negative definite function on \( \mathbb{R}^7 \).

Thus, by Lyapunov stability theory [30], it is immediate that the controlled Cai system (7) is globally exponentially stable and also that the parameter estimation errors \( e_a, e_b, e_c, e_d \) exponentially converge to zero with time.

This completes the proof. \( \blacksquare \)

3.2 Numerical Results

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the chaotic system (3) with the adaptive control law (4) and the parameter update law (11).

The parameters of the Cai system (3) are selected as

\[
a = 20, \ b = 14, \ c = 10.6 \ \text{and} \ d = 2.8
\]

For the adaptive and update laws, we take

\[
k_i = 5, \ (i = 1, 2, \ldots, 7).
\]

Suppose that the initial values of the estimated parameters are

\[
\hat{a}(0) = 9, \ \hat{b}(0) = 22, \ \hat{c}(0) = 14, \ \hat{d}(0) = 5
\]

The initial state of the controlled Cai system (3) is taken as

\[
x_1(0) = 9, \ x_2(0) = 24, \ x_3(0) = -20
\]

When the adaptive control law (4) and the parameter update law (11) are used, the controlled modified Cai system converges to the equilibrium \( E_0 = (0, 0, 0) \) exponentially as shown in Figure 2.

The time-history of the parameter estimates is shown in Figure 3.

The time-history of the parameter estimation errors is shown in Figure 4.
Figure 2. Time Responses of the Controlled Cai System

Figure 3. Time-History of the Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t)$
4. ADAPTIVE SYNCHRONIZATION OF IDENTICAL CAI CHAOTIC SYSTEMS

4.1 Theoretical Results

In this section, we discuss the adaptive synchronization of identical Cai systems (2007) with unknown parameters.

As the master system, we consider the Cai dynamics described by

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= bx_1 + cx_2 - x_1x_3 \\
\dot{x}_3 &= x_1^2 - dx_3
\end{align*}
\]  

(13)

where \( x_i \), \( i = 1, 2, 3 \) are the state variables and \( a, b, c, d \) are unknown system parameters.

As the slave system, we consider the controlled Cai system described by

\[
\begin{align*}
\dot{y}_1 &= a(y_2 - y_1) + u_1 \\
\dot{y}_2 &= by_1 + cy_2 - y_1y_3 + u_2 \\
\dot{y}_3 &= y_1^2 - dy_3 + u_3
\end{align*}
\]  

(14)
where \( y_i, (i = 1, 2, 3) \) are the state variables and \( u_i, (i = 1, 2, 3) \) are adaptive controllers to be designed.

The synchronization error is defined by

\[
\begin{align*}
e_1 &= y_1 - x_1 \\
e_2 &= y_2 - x_2 \\
e_3 &= y_3 - x_3
\end{align*}
\]

(15)

Then the error dynamics is obtained as

\[
\begin{align*}
\dot{e}_1 &= a(e_2 - e_1) + u_1 \\
\dot{e}_2 &= b e_1 + c e_2 - y_1 y_3 + x_1 x_3 + u_2 \\
\dot{e}_3 &= -d e_3 + y_1^2 - x_1^2 + u_3
\end{align*}
\]

(16)

Let us now define the adaptive control functions \( u_1(t), u_2(t), u_3(t) \) as

\[
\begin{align*}
u_1 &= -\hat{a}(e_2 - e_1) - k_e e_1 \\
u_2 &= -\hat{b} e_1 - \hat{c} e_2 + y_1 y_3 - x_1 x_3 - k_e e_2 \\
u_3 &= \hat{d} e_3 - y_1^2 + x_1^2 - k_e e_3
\end{align*}
\]

(17)

where \( \hat{a}, \hat{b}, \hat{c}, \) and \( \hat{d} \) are estimates of the parameters \( a, b, c \) and \( d \), respectively, and \( k_e, (i = 1, 2, 3) \) are positive constants.

Substituting the control law (17) into (16), we obtain the error dynamics as

\[
\begin{align*}
\dot{e}_1 &= (a - \hat{a})(e_2 - e_1) - k_e e_1 \\
\dot{e}_2 &= (b - \hat{b}) e_1 + (c - \hat{c}) e_2 - k_e e_2 \\
\dot{e}_3 &= -(d - \hat{d}) e_3 - k_e e_3
\end{align*}
\]

(18)

Let us now define the parameter errors as

\[
\begin{align*}
e_a &= a - \hat{a}, \quad e_b = b - \hat{b}, \quad e_c = c - \hat{c}, \quad e_d = d - \hat{d}
\end{align*}
\]

(19)

Substituting (19) into (18), the error dynamics simplifies to

\[
\begin{align*}
\dot{e}_1 &= e_a (e_2 - e_1) - k_e e_1 \\
\dot{e}_2 &= e_b e_1 + e_c e_2 - k_e e_2 \\
\dot{e}_3 &= -e_d e_3 - k_e e_3
\end{align*}
\]

(20)

Consider the quadratic Lyapunov function
\[ V(e_1, e_2, e_3, e_a, e_b, e_c, e_d) = \frac{1}{2} \left( e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2 \right), \]  
(21)

which is a positive definite function on \( \mathbb{R}^7 \).

Note also that
\[
\dot{\hat{a}} = -\hat{a}, \quad \dot{\hat{b}} = -\hat{b}, \quad \dot{\hat{c}} = -\hat{c}, \quad \dot{\hat{d}} = -\hat{d}
\]
(22)

Differentiating \( V \) along the trajectories of (20) and using (22), we obtain
\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 + e_a \left[ e_1 (e_2 - e_1) - \hat{a} \right] + e_b \left( e_1 e_2 - \hat{b} \right) 
+ e_c \left( e_2^2 - \hat{c} \right) + e_d \left( -e_3^2 - \hat{d} \right)
\]
(23)

In view of Eq. (23), the estimated parameters are updated by the following law:
\[
\hat{a} = e_1 (e_2 - e_1) + k_4 e_a \\
\hat{b} = e_1 e_2 + k_5 e_b \\
\hat{c} = e_2^2 + k_6 e_c \\
\hat{d} = -e_3^2 + k_7 e_d 
\]
(24)

where \( k_4, k_5, k_6 \) and \( k_7 \) are positive constants.

**Theorem 2.** The identical Cai systems (13) and (14) with unknown parameters are globally and exponentially synchronized for all initial conditions by the adaptive control law (17), where the update law for parameters is given by (24) and \( k_i, (i = 1, \ldots, 7) \) are positive constants.

**Proof.** Substituting (24) into (23), we get
\[
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_a^2 - k_5 e_b^2 - k_6 e_c^2 - k_7 e_d^2 
\]
(25)

From (25), we find that \( \dot{V} \) is a negative definite function on \( \mathbb{R}^6 \).

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error and the parameter error decay to zero exponentially with time for all initial conditions. ■

**4.2 Numerical Results**

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the two systems of differential equations (13) and (14) with the adaptive control law (17) and the parameter update law (24).

We take the parameter values as in the chaotic case, viz.
We take the positive constants $k_i, \ (i=1,\ldots,7)$ as

$$k_i = 5 \quad \text{for} \quad i=1,2,\ldots,7.$$ 

Suppose that the initial values of the estimated parameters are

$$\hat{a}(0) = 16, \quad \hat{b}(0) = 4, \quad \hat{c}(0) = 8, \quad \hat{d}(0) = 5.$$ 

We take the initial values of the master system (13) as

$$x_1(0) = -5, \quad x_2(0) = 25, \quad x_3(0) = 14.$$ 

We take the initial values of the slave system (14) as

$$y_1(0) = 24, \quad y_2(0) = 10, \quad y_3(0) = -9.$$ 

Figure 5 shows the adaptive chaos synchronization of the identical Cai systems.

Figure 6 shows the time-history of the synchronization error $e_1, e_2, e_3$.

Figure 7 shows the time-history of the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t)$.

Figure 8 shows the time-history of the parameter estimation errors $e_a, e_b, e_c, e_d$.

Figure 5. Adaptive Synchronization of the Cai Systems
Figure 6. Time-History of the Synchronization Error $e_1, e_2, e_3$

Figure 7. Time-History of the Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$
5. CONCLUSIONS

In this paper, we derived new results for the design of adaptive control and synchronization of the Cai system (2007) with unknown system parameters. First, we designed adaptive control laws to stabilize the Cai system to its unstable equilibrium point at the origin based on the adaptive control theory and Lyapunov stability theory. Then we derived adaptive synchronization scheme and update law for the estimation of system parameters for the identical Cai systems with unknown parameters. Our synchronization schemes were established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the proposed adaptive control method is very effective and convenient to achieve chaos control and synchronization of the Cai system. Numerical simulations are shown to depict the effectiveness of the proposed adaptive control and synchronization schemes.

REFERENCES


Author

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