Global Stabilization of a Class of Nonlinear System Based on Reduced-order State Feedback Control

Chang-Zhong Chen¹,², Tao Fan¹,², Bang-Rong Wang³, Dong-Ming Xie¹,² and Ping He¹,²

¹School of Automation and Electronic Information, Sichuan University of Science & Engineering, Zigong, Sichuan 643000, People’s Republic of China; ²Artificial Intelligence Key Laboratory of Sichuan Province, Sichuan University of Science & Engineering, Zigong, Sichuan 643000, People’s Republic of China; ³Department of Mathematics, School of Science, Sichuan University of Science & Engineering, Zigong, Sichuan 643000, People’s Republic of China.

ABSTRACT

The problem of global stabilization for a class of nonlinear system is considered in this paper. The sufficient condition of the global stabilization of this class of system is obtained by deducing the stabilization of itself from the stabilization of its subsystems. This paper will come up with a design method of state feedback control law to make this class of nonlinear system stable, and indicate the efficiency of the conclusion of this paper via a series of examples and simulations at the end. The results presented in this paper improve and generalize the corresponding results of recent works.

KEYWORDS

Nonlinear systems; Reduced-order control; Global stabilization.

1. Introduction

Stabilization is a seriously important topic in control system designing [1]-[13]. In recent years the stabilization of nonlinear systems has won extensive attention from researchers, and some achievements have been made. The problem is simplified into the question of stabilization of low order systems, the sufficient condition of the system is gained, and that the state feedback control law is designed by combining centre manifold theory with part feedback linearization (e.g., [14]).

The design method to construct global stabilization feedback control law is founded and several sufficient condition of global stabilization for the system is gained by applying linearization method to a class of nonlinear systems (e.g., [15, 16]). The sufficient condition of local stabilization and global stabilization for the system is gained by researching the stabilization question of minimum phase nonlinear system (e.g., [16]). A new method to design control laws is given, and it has been proved that the corresponding close-loop system can be globally stable under appropriate condition by researching the stabilization question of a class of affine nonlinear
system with standard form (e.g., [17]). The sufficient condition for the global stabilization of several smooth feedback is given by researching the global stabilization of cascade systems constructed with linear systems which can be made stable and asymptotically stable nonlinear systems (e.g., [18]). The stabilization question of a class of triangle system is researched with smooth output feedback (e.g., [19]). The global stabilization question of cascade system made up of two nonlinear systems is researched via constructing Lyapunov function (e.g., [20]). The sufficient condition for the system to be stable is gained, and a state feedback control law is designed out through researching the stabilization question of a class of non-minimum phase-nonlinear system based on drive control (e.g., [22]).

This paper will research the global stabilization question of a class of nonlinear system on the base of analysis, (e.g. See Ref. [8]-[12]), and deduce the stabilization of the original system from the stabilization of its subsystem-reduced-order control system. According to the features of this class of nonlinear system, this paper will come up with a design method for the corresponding state feedback control law, and will prove that close-loop system is globally asymptotically stable under appropriate condition with Lyapunov second method. This paper will indicate the efficiency of the conclusion of this paper via a series of examples and simulations at the end. In this paper, if not specially illustrated, $k \cdot k$ refers to Euclid norm, $k \cdot k_F$ refers to Frobenius norm, $| \cdot |$ refers to scalar function or absolute value of function.

2. System analysis

Consider the following nonlinear system

\[
\begin{cases}
\dot{x} = f(x) + 2h(x, y)y, \\
\dot{y} = u + g(x, y)y.
\end{cases}
\]

(1)

where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^{n-m}$ are state vectors, $u \in \mathbb{R}^{n-m}$ is a input vector. $f(x)$ is a vector function of $x$, $h(x, y)$ is a vector function of $(x, y)$, $g(x, y)$ is a scalar function of $(x, y)$, and $f(x)$, $g(x, y)$ and $h(x, y)$ are at least $C^1$ functions about $x$ or $(x, y)$, respectively.

Assumption 1. The subsystem of system (1)

\[
\dot{x} = f(x).
\]

(2)

is globally stable.

Remark 1. This paper uses the static stability of system (1) to discuss its dynamic stability. The purpose of Assumption 1 is to ensure the static stability of system (1).

In order to obtain main result of this paper, we should do further supposition.

Assumption 2. The subsystem (2) of system (1) is globally stable and there is a Lyapunov
function \( v(x) \) which satisfies

\[
\dot{v}(x) \big|_{(2)} = \frac{\partial v(x)}{\partial x} f(x) = \left[ \frac{\partial v(x)}{\partial x_1} \frac{\partial v(x)}{\partial x_2} \ldots \frac{\partial v(x)}{\partial x_m} \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix} \\
\leq -1, \forall \| x \| \geq M.
\]

where \( M \) is a positive constant.

Before proving main theorems, present the following analysis and *Lemma 1* at first. Consider the following nonlinear system

\[
\begin{align*}
\dot{x} &= f(x, y), \\
\dot{y} &= -yg(x, y).
\end{align*}
\]

where \( x \in \mathbb{R}^m \) and \( y \in \mathbb{R}^{n-m} \) are state vectors, \( f(x, y) \) and \( g(x, y) \) are vector function of \( (x, y) \), and \( f(x, y) \) and \( g(x, y) \) are at least \( C^1 \) functions about \( (x, y) \).

**Lemma 1.** (e.g. see Ref. [23]) If the subsystem of system (3)

\[
\dot{x} = f(x, 0).
\]

is globally stable, and there is a continuous differentiable scalar functions \( v(x) \) such that \( \lim_{x \to 0} v(x) = +\infty \), and the derivative of along the trajectory of system (4) at \( k x k > 0 \) such that \( v(x) \leq 0 \), then system (3) is globally stable.

**Remark 2.** *Lemma 1* obtained is rather significant for the proving of the main theorems in this paper, the main idea of which is that the stabilization of the original system can be proved when reduced-order subsystem such that static stabilization and some essential assumed conditions. Before the theorem of this paper given, we must give the following *Lemma 2* at first, which is is easy to understand.

**Lemma 2.** (e.g. see Ref. [26], [27]) Assume that subsystem (2) of nonlinear system (1) is globally asymptotically stable, then from *Converse-Lyapunov theorem*, we know that there must be a positive function \( v(x) \) which satisfies

\[
\dot{v}(x) \big|_{(2)} = \frac{\partial v(x)}{\partial x} f(x) = \left[ \frac{\partial v(x)}{\partial x_1} \frac{\partial v(x)}{\partial x_2} \ldots \frac{\partial v(x)}{\partial x_m} \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}^T \\
\leq 0, \forall x \in \mathbb{R}^m - \{0, 0, \ldots, 0\}.
\]

(5)
Without loss of generality. We can suppose
\[ \dot{v}(x) \leq -\varepsilon < 0, \forall x \in \mathbb{R}^n - \{0, 0, \ldots, 0\}. \]

where " is a very small positive.

3. Main results

In the process of researching nonlinear systems, it’s always desired that the states of the system can reach stable values, so that as many interferences as possible could be avoided. Therefore, making systems stable has a strong practical applicability. Because of that this question has a certain actual application background, the research in this paper has some practical meaning.

3.1 Reduced-order state feedback controller design

**Theorem 1.** Assume that subsystem (2) of nonlinear system (1) satisfies Assumption 2 and there is a scalar function \( K(x, y) \) for system (1) which satisfies:

\[ K(x, y) > \left\| \begin{bmatrix} \frac{\partial u(x)}{\partial x_1} & \frac{\partial u(x)}{\partial x_2} & \cdots & \frac{\partial u(x)}{\partial x_m} \end{bmatrix} \right\|_2 \cdot \| h(x, y) \|_2. \]  

(6)

Then there is a state feedback control low

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_{n-m}
\end{bmatrix} = \begin{bmatrix}
  -y_1 K(x, y) \\
  -y_2 K(x, y) \\
  \vdots \\
  -y_{n-m} K(x, y)
\end{bmatrix} - \begin{bmatrix}
  y_1 g(x, y) \\
  y_2 g(x, y) \\
  \vdots \\
  y_{n-m} g(x, y)
\end{bmatrix}
\]  

(7)

such that the system (1) globally asymptotically stable, where the elements of \( K(x, y) \) are at least \( C_1 \) functions about \( (x, y) \).

**Proof.** After substitution the state feedback control low (7) into system (1) we see that

\[
\begin{cases}
  \dot{x} = f(x) + 2h(x, y)y, \\
  \dot{y} = -yK(x, y).
\end{cases}
\]  

(8)

According to Lemma 2, we can construct a Lyapunov function

\[ V(x, y) = v(x) + \frac{1}{2} y^Ty. \]
then its derivative along the trajectory of system (1) is

\[
\dot{V}(x, y) |_{(1)} = \left[ \frac{\partial v(x)}{\partial x_1} \frac{\partial v(x)}{\partial x_2} \ldots \frac{\partial v(x)}{\partial x_m} \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}
+ 2\left[ \frac{\partial v(x)}{\partial x_1} \frac{\partial v(x)}{\partial x_2} \ldots \frac{\partial v(x)}{\partial x_m} \right] \cdot h(x, y) y + \| y \|^2 - M \cdot [K(x, y)]
\leq 0 + \| \left[ \frac{\partial v(x)}{\partial x_1} \frac{\partial v(x)}{\partial x_2} \ldots \frac{\partial v(x)}{\partial x_m} \right] \cdot h(x, y) \|_F \cdot \| y \| + \| y \|^2 - M \cdot [K(x, y)].
\]

Combined with condition (6) we easily know

\[
\dot{V}(x) < 0, \forall x \in \mathbb{R}^m - \{0, 0, \ldots, 0\}.
\]

Then we can conclude from Lemma 1 and analysis, e.g. see Ref. 8-12, that system (1) is globally asymptotically stable.

### 3.2 Further result in mathematics

The condition of subsystem (2) respect to system (1) is weakens and the further investigation is given.

**Theorem 2.** Assume that subsystem (2) of nonlinear system (1) satisfies Assumption 1. Then there must be a positive \( M \), when \( K(x, y) + g(x, y) < -M \), there is a state feedback control low

\[
\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-m} \end{bmatrix} = \begin{bmatrix} y_1 K(x, y) \\ y_2 K(x, y) \\ \vdots \\ y_{n-m} K(x, y) \end{bmatrix}
\]

such that the system (1) globally asymptotically stable, where the elements of \( K(x, y) \) are at least \( C^1 \) functions about \( (x, y) \).

**Proof.** After substitution the state feedback control low (9) into system (1) we see that

\[
\begin{cases}
\dot{x} = f(x) + 2h(x, y)y, \\
\dot{y} = yK(x, y) + g(x, y)y.
\end{cases}
\]
According to Lemma 2, we can construct a Lyapunov function

\[ V(x, y) = v(x) + \frac{1}{2} y^T y. \]

then its derivative along the trajectory of system (1) is

\[
\dot{V}(x, y) \bigg|_{(1)} = \left[ \frac{\partial v(x)}{\partial x_1} \frac{\partial v(x)}{\partial x_2} \cdots \frac{\partial v(x)}{\partial x_m} \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix} \\
-\varepsilon + 2 \left[ \frac{\partial v(x)}{\partial x_1} \frac{\partial v(x)}{\partial x_2} \cdots \frac{\partial v(x)}{\partial x_m} \right] \cdot h(x, y) y + \| y \|^2 \cdot [K(x, y) + g(x, y)] \\
\leq 0 + \| \frac{\partial v(x)}{\partial x_1} \frac{\partial v(x)}{\partial x_2} \cdots \frac{\partial v(x)}{\partial x_m} \| \cdot \| h(x, y) \|_F \cdot \| y \| \\
+ \| y \|^2 \cdot [K(x, y) + g(x, y)].
\]

Combined with condition \( K(x, y) + g(x, y) < -M \) we easy to know that so long as

\[ M > \frac{\| \frac{\partial v(x)}{\partial x_1} \frac{\partial v(x)}{\partial x_2} \cdots \frac{\partial v(x)}{\partial x_m} \|_2 \cdot \| h(x, y) \|_F^2}{\varepsilon} \]

there must be

\[ \dot{V}(x) < 0, \forall x \in \mathbb{R}^m - \{0, 0, \cdots, 0\}. \]

Then we can conclude from Lemma 1 and analysis, e.g. see Ref. 8-12, that system (1) is globally asymptotically stable.

4. System simulations

In this section, in order to show that the approach of this paper to this sort of control system is effective and convenient, we give some illustrative examples and simulation results.

Example 1. Consider the following second-order nonlinear system

\[
\begin{cases}
\dot{x} = -x^3 + 2y^2, \\
\dot{y} = u + 3xy^2 + y^2.
\end{cases}
\]

(11)
**Solution.** Base on the theorem above, for the subsystem of system (11), we can construct a Lyapunov function which is given by

\[ v(x) = x^2. \]

It is easy to know that the subsystem of system (11)

\[ \dot{x} = -x^3. \]

is globally asymptotically stable.

Base on the condition (6) of *Theorem 1*, let

\[ K = 4x^2y^2. \]

Then, there is a state feedback control law

\[ u = -4x^2y^3 - 3xy^2 - y^2. \] \hspace{1cm} (12)

such that the system (11) is globally asymptotically stabilization.

**Simulation.** Without loss of generality. Let the initial values of simulation

\[ \begin{cases} x(0) = -4, \\ y(0) = +3. \end{cases} \]

and the input control signal

\[ u = \begin{cases} 0.5, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases} \]

The sample time is 0.1s, and the simulation time are 50s seconds. The dynamic response of the system (11) with the state feedback control law (12) is show in Figure 1.
From the Figure 1, we can know that the controlled closed-loop system of the system (11) is asymptotically stable.

**Example 2.** Consider the following third-order nonlinear system

\[
\begin{align*}
\dot{x} &= -x^3 + 2y^2, \\
\dot{y}_1 &= u_1 + xy_1^2 + xy_1y_2 + xy_1y_3, \\
\dot{y}_2 &= u_2 + xy_1y_2 + xy_2^2 + xy_2^4.
\end{align*}
\]  

(13)
Solution: Base on the theorem above, for the subsystem of system (13), we can construct a Lyapunov function which is given by

\[ v(x) = \frac{1}{2} x^2. \]

It is easy to know that the subsystem of system (13)

\[ \dot{x} = -x. \]

is globally asymptotically stable.

Base on the condition (6) of Theorem 1, let

\[ K = x^2(2y_1^2 + 2y_2^2) + 1. \]

Then, there is a state feedback control law

\[
\begin{align*}
    u_1 &= -2x^2y_1^3 - 2x^2y_1y_2^2 - y_1 - xy_1^2 - xy_1y_2 - xy_1y_2^3, \\
    u_2 &= -2x^2y_1^2y_2 - 2x^2y_2^3 - y_2 - xy_1y_2 - xy_2^3 - xy_2^4.
\end{align*}
\]

(14)

such that the system (13) is globally asymptotically stabilization.

Simulation. Without loss of generality. Let the initial values of simulation

\[
\begin{align*}
    x(0) &= 2, \\
    y_1(0) &= -5, \\
    y_2(0) &= 4.
\end{align*}
\]

and the input control signal

\[
u = \begin{cases} 
1, & \text{for } t \geq 0, \\
0, & \text{for } t < 0.
\end{cases}
\]

The sample time is 0.1s, and the simulation time are 10s seconds.

The dynamic response of the system (13) with the state feedback control law (14) is show in Figure 2.

From the Figure 2, we can know that the controlled closed-loop system of the system (13) is asymptotically stable [28].
5. Conclusion

Based on some work of pioneers, the global stabilization problem for a class of nonlinear system is investigated in this paper, and that the stabilization of the original system is obtained from the stabilization of its reduced-order subsystem. A design method for the corresponding state feedback control law is given according to the features of this class of nonlinear system, and that the close-loop system is globally asymptotically stable under appropriate condition is proved with Lyapunov second method. This paper has indicated the efficiency of the conclusion of this paper via a series of examples and simulations at the end.

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References


Authors

**Chang-Zhong Chen** received the B.S. Degree in Automatic Control from Sichuan University, in Chengdu, Sichuan, People’s Republic of China and the M.S. Degrees in Control Science and Engineering from Sichuan University of Science & Engineering, in Zigong, Sichuan, China, in 1997 and 2009, respectively. He is currently an Associate Professor with the School of Automation and Electronic Information, Sichuan University of Science & Engineering, in Zigong, Sichuan, China, from 2010. His research interests include Technology and application of microcontrollers, Intelligent control and Chaos control.

**Tao Fan** received the B.S. Degree and the M.S. Degree in Measurement Technology and Automation Device from Beijing University of Chemical Technology, in Beijing, People’s Republic of China, in 2003 and 2006, respectively. He is currently a Lecturer with the School of Automation and Electronic Information, Sichuan University of Science & Engineering, in Zigong, Sichuan, China. His main research interests include Intelligent control, Chaos control and synchronization, Complex networks and Consensus of multi-agent systems.