THE RESEARCH OF QUANTUM PHASE ESTIMATION ALGORITHM

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ABSTRACT

A quantum computation problem is discussed in this paper. Many new features that make quantum computation superior to classical computation can be attributed to quantum coherence effect, which depends on the phase of quantum coherent state. Quantum Fourier transform algorithm, the most commonly used algorithm, is introduced. And one of its most important applications, phase estimation of quantum state based on quantum Fourier transform, is presented in details. The flow of phase estimation algorithm and the quantum circuit model are shown. And the error of the output phase value, as well as the probability of measurement, is analysed. The probability distribution of the measuring result of phase value is presented and the computational efficiency is discussed. The researchis focused on how to determine the optimum number of qubit which satisfy the accuracy of phase estimation.

KEYWORDS

Quantum computation, Quantum Fourier transform, Phase estimation

1. INTRODUCTION

In the past few decades, we have gained more and more ability to access massiveamounts of information and to make use of computers to store, analyse and manage thisdata. Recent studies have shown the great progress towards the physical realization of a quantum computer. Quantum computation and quantum information is the study of the information processing task that can be accomplished using quantum mechanical system. The process of quantum computation is the evolution process ofquantum mechanical system. Quantum computation has many features that differ from classical computing in that quantum state has the characteristic of coherence and entanglement.

Deutsch[1] [2]systematically described the first universal quantum computer that is accepted by now.In 1982, Benioff[3]studied the question whether quantum computer was more computationally powerful than a conventional classical Turing machine. He mapped the operation of a reversible Turing machine onto thequantum system and thus exhibited the first quantum-mechanical model of computation, which discovered the potential power of quantum computer. In 1994, American scientist Peter Shor[4] proposed an algorithm that factor a large integer inpolynomial time, which is the first practical quantum algorithm. In 1996, Grover[5] [6] proposed an algorithm that provides a speedup of $\overline{N}$ in order of magnitude than classic algorithm in searching an unsorted database. The algorithmcaused attention as its potential of solving NP problem.

2. THE QUANTUM FOURIER TRANSFORM (QFT)

One of the most useful methods of solving problems in mathematics or computer science is to transform it into some other problem for which a solution is known. The quantum Fourier transform[7] is a kind of discrete Fourier transform which plays an important role in many quantum algorithms. For example, Shor's algorithm, order finding algorithm and hidden subgroup problem.

Quantum Fourier Transform is defined as follow:

$$U_{QFT}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{\frac{2\pi i xy}{2^n}}|y\rangle$$

The QFT is a unitary operator in 2^n-dimensional vector space, it is defined as a linear operator in a group of orthonormal base $|0\rangle, |1\rangle\cdots|2^n-1\rangle$.

Since any quantum computation algorithm in an n-qubit quantum computer is based on operations by matrices in $U(2^n$-dimensional), in this sense we have the universality of the QFT.In the QFT, we used the following basic quantum gates: Hadmard gate and Controlled-Phase Shift gate. Hadmard gate(H) is acts on one qubit. Controlled-Phase Shift gate($U_{j,k}$) is acts on two qubit, k is control qubit and j is target qubit. Applying the unitary transformation $U_{j,k}$ on the j[th]qubit if and only if the k[th]qubit is $|1\rangle$. The transform matrixes of the gates operator in Hilbert space are presented as follow: (Phase shift $\theta_{j,k} = \frac{2\pi x_k}{2^{k-j+1}}$)

$$H = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \quad U_{j,k} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta_{j,k}} \end{vmatrix}$$

Phase shift $\theta_{j,k} = \frac{2\pi x_k}{2^{k-j+1}}$

Hadmard gate and Controlled-Phase Shift gate have the following mapping relationships when they act on any qubits$|x\rangle$ or $|x\rangle|y\rangle$:

$$|x\rangle \xrightarrow{H} (-1)^x|x\rangle + |1-x\rangle \qquad\qquad |x\rangle|y\rangle \xrightarrow{U_{j,k}} e^{\frac{2\pi i xy}{2^{k-j}}}|x\rangle|y\rangle$$

The QFT can be given the following useful product representation:

$$U_{QFT}|x_1\cdots x_{n-1}x_n\rangle = \frac{1}{\sqrt{2^n}}(|0\rangle + e^{2\pi i 0.x_n}|1\rangle)\cdots(|0\rangle + e^{2\pi i 0.x_1\cdots x_{n-1}x_n}|1\rangle)$$

Most of the properties of the quantum Fourier transform follow from the fact that it is a unitary transformation. From the unitary property it follows that the inverse of the quantum Fourier transform is the Hermitianadjoint of the Fourier matrix, therefore,$U_{QFT}^{-1} = U_{QFT}^{\dagger}$. Since there is an efficient quantum circuit implementing the quantum Fourier transform, the circuit can be run in reverse to perform the inverse quantum Fourier transform. Thus both transforms can be efficiently performed on a quantum computer.

The inverse of QFT is：

$$U_{QFT}{}^{\dagger}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-\frac{2\pi i \cdot xy}{2^n}} |y\rangle$$

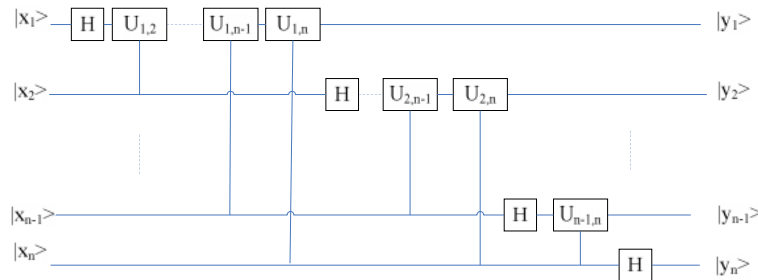The quantum circuit representation of the QFT is show in Figure1.

Figure 1.Circuit of the quantum Fourier transform

3. PHASE ESTIMATION

Suppose, there is an unitary operator has an eigenvector $|a\rangle$ with eigenvalue $e^{2\pi i \varphi}$ (n-qubit),$U|a\rangle = e^{2\pi i \varphi}|a\rangle$. The quantum phase estimation[8] procedure uses two registers. The first register with m-qubit , it used to store the value of phase $|\varphi_1 \varphi_2 \cdots \varphi_m \rangle$ ($\varphi_i = 0$ or 1， i = 1,2 … m), in decimal it equals to $\sum_{i=1}^{m} \frac{\varphi_i}{2^i}$. The error of true value and store value is μ which can be expressed: $\varphi = (\sum_{i=1}^{m} \frac{\varphi_i}{2^i}) + \mu$ ($|\mu| \leq \frac{1}{2^{m+1}}$).The first register initially in the state $|0 \cdots 0\rangle$ while the second is$|a\rangle$.Apply the quantum circuit shown in Figure 2. ($U_n = U^{2^n}$)
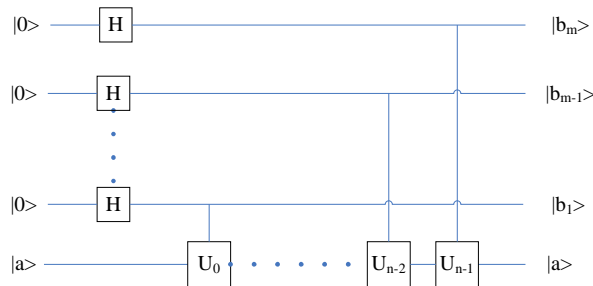

Figure 2.First stage of the phase estimation

Apply the inverse quantum Fourier transform on the first register ($|b_1 \cdots b_{m-1}b_m\rangle$), we have the state:

$$\frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} e^{\frac{-2\pi i jk}{2^n}} e^{2\pi i \varphi j}|k\rangle$$

$$\frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} e^{\frac{-2\pi i j}{2^n}(k-\varphi\cdot 2^n)} |k\rangle$$

And then, read out the state of first register by doing a measurement in the computation basis, the result is that $\widetilde{\varphi} = 0.\, {}_1\cdots {}_m$ which is an estimator for .

We should give the accuracy ( ) of phase estimation, when $k - \varphi\cdot 2^n$ $0\ (k = 0,1\cdots 2^n - 1)$. The state of first register must meet:

$$|k_l - \varphi\cdot 2^n| \quad \varepsilon,\ (l = a\cdots b)$$

The probability of phase estimation which satisfied the accuracy is:

$$P = \sum_{l=a}^{b} \left|\frac{1}{2^n} \sum_{j=0}^{2^n-1} e^{-\frac{2\pi i\cdot j}{2^n}(k_l-\varphi\cdot 2^n)}\right|^2$$

The accuracy of the first register must be lessthan $,\frac{1}{2^{n+1}}$. So, n $log_2\frac{1}{\varepsilon} - 1$ (n is an integer). It will spend $n(n + 5)/2$ quantum logic gates when take a phase estimation operation. The qualified phase will be measured at least once and the expectation of quantum logic gates is:

$$EN(n) = \frac{n(n + 5)}{2\cdot \sum_{l=a}^{b}\left|\frac{1}{2^n}\sum_{j=0}^{2^n-1} e^{-\frac{2\pi i\cdot j}{2^n}(k_l-\varphi\cdot 2^n)}\right|^2}$$

n $log_2\frac{1}{\varepsilon} - 1$, n is an integer.

The minimum of $EN(n)$ may be at the point $\frac{\partial EN(n)}{\partial n} = 0$ or $n = n_\alpha$ ($n_\alpha$is a minimum integer that n_α $log_2\frac{1}{\varepsilon} - 1$).

If the equation $\frac{\partial EN(n)}{\partial n} = 0$ has solution in its domain. Suppose the solution is n_β. If n_β is an integer. The minimum of $EN(n)$is the one point of $n_\alpha\ and\ n_\beta$, which satisfy:

$$EN(n_\gamma) = min\ (EN(n_\alpha), EN(n_\beta))$$

If n_β is not an integer. The minimum of $EN(n)$ is the one point of $n_\alpha, n_{\beta 1}$and $n_{\beta 2}$, which satisfy: ($n_{\beta 1}, n_{\beta 2}$ are integers, and $n_{\beta 1} < n_\beta < n_{\beta 2}$)

$$EN(n_\gamma) = min\ (EN(n_\alpha), EN(n_{\beta 1}), EN(n_{\beta 2}))$$

4. SIMULATION

We proposed a quantum computationemulator[9][10] in classical computer which satisfies the requirements of quantum computer. Conventional quantum algorithms can be operatedin this simulation system. In this paper, the simulation aimed to analyses the relationship between the accuracy and probability of observing in phase estimation algorithm. The phase can be written in

n-qubitin the phase estimation algorithm. We will get all the possible states by measuring the output qubit.

4.1. Probability distribution of Phase Estimation

In the first simulation, we provide 8-qubit for the phase estimation. It means the minimum accuracy is 1/256. Suppose the phase is 1/3. We can get the probability distributions of quantum state of output from $|0 \cdots 0\rangle$ to $|1 \cdots 1\rangle$which is shown in Figure 3.
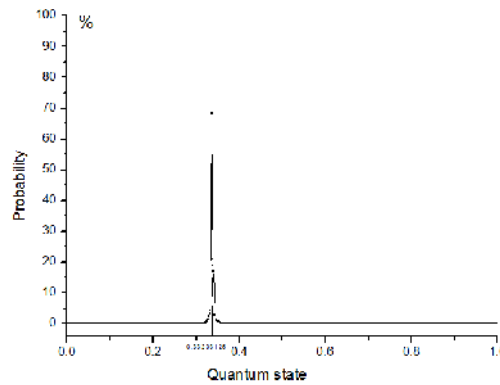

Figure 3.The probability distribution of phase estimation

4.2. The number of qubit effect on computational performance

The second simulation is mainly about the different numbers of qubit effect on the probability of phase estimation which in the same accuracy. Suppose the phase is 1/3 and we wish to approximate the phase to an accuracy 1/64. And then we provide different number of qubit for the phase estimation. In Figure 4, (a) is the probability of success estimation by different number of qubit and (b) is the time steps by different number of qubit.
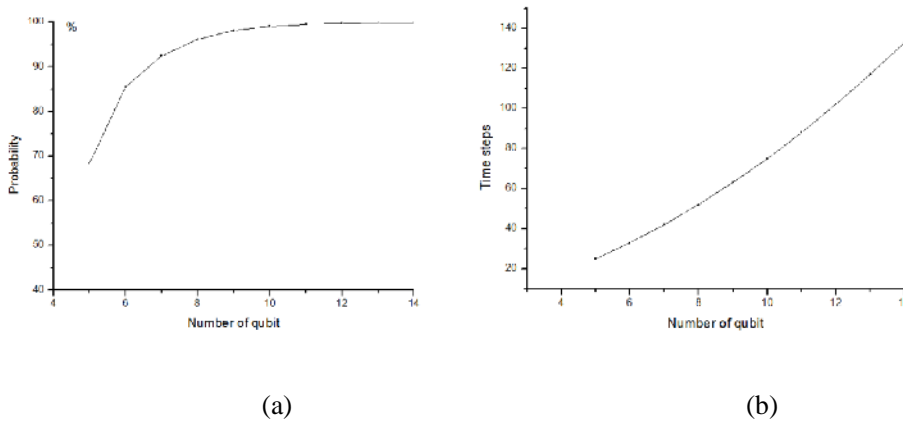

(a)   (b)

Figure 4.The probability and time steps by different number of qubit

5. CONCLUSIONS

This paper describes the phase estimation algorithm based in quantum Fourier transform.And the algorithm was operated in a simulation system. In Figure3, the result converged to the expectation value and it proved the phase estimation algorithm is an efficient estimate. In Figure4, we can see the probability increased with the number of qubit with given accuracy, but the probability changes slowly when the number of qubit reaches a certain value. Meanwhile, the time steps were increased in polynomial multiple. We should design appropriate number of qubit for the phase estimation with high success probability and low cost. The optimum number of qubit has been given which satisfy the accuracy after above analysis on phase estimation. In future, our study focuses on the optimization methods in quantum computation.

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