# ANTI-SYNCHRONIZATION OF HYPERCHAOTIC WANG AND HYPERCHAOTIC LI SYSTEMS WITH UNKNOWN PARAMETERS VIA ADAPTIVE CONTROL

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***ABSTRACT***

*In chaos theory, the problem anti-synchronization of chaotic systems deals with a pair of chaotic systems called drive and response systems. In this problem, the design goal is to drive the sum of their respective states to zero asymptotically. This problem gets even more complicated and requires special attention when the parameters of the drive and response systems are unknown. This paper uses adaptive control theory and Lyapunov stability theory to derive new results for the anti-synchronization of hyperchaotic Wang system (2008) and hyperchaotic Li system (2005) with uncertain parameters. Hyperchaotic systems are nonlinear dynamical systems exhibiting chaotic behaviour with two or more positive Lyapunov exponents. The hyperchaotic systems have applications in areas like oscillators, lasers, neural networks, encryption, secure transmission and secure communication. The main results derived in this paper are validated and demonstrated with MATLAB simulations.*

***KEYWORDS***

*Hyperchaos, Hyperchaotic Systems, Adaptive Control, Anti-Synchronization.*

## 1. INTRODUCTION

Hyperchaotic systems are typically defined as nonlinear chaotic systems having two or more positive Lyapunov exponents. They are applicable in several areas like lasers [1], chemical reactions [2], neural networks [3], oscillators [4], data encryption [5], secure communication [6-8], etc.

In chaos theory, the anti-synchronization problem deals with a pair of chaotic systems called the *drive* and *response* systems, where the design goal is to render the respective states to be same in magnitude, but opposite in sign, or in other words, to drive the sum of the respective states to zero asymptotically [9].

There are several methods available in the literature to tackle the problem of synchronization and anti-synchronization of chaotic systems like active control method [10-12], adaptive control method [13-15], backstepping method [16-19], sliding control method [20-22] etc.

This paper derives new results for the adaptive controller design for the anti-synchronization of hyperchaotic Wang systems ([23], 2008) and hyperchaotic Li systems ([24], 2005) with unknown parameters. Lyapunov stability theory [25] has been applied to prove the main results of this paper. Numerical simulations have been shown using MATLAB to illustrate the results.

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## 2. THE PROBLEM OF ANTI-SYNCHRONIZATION OF CHAOTIC SYSTEMS

In chaos synchronization problem, the *drive system* is described by the chaotic dynamics

$$\dot{x} = Ax + f(x) \tag{1}$$

where $A$ is the $n \times n$ matrix of the system parameters and $f : R^n \to R^n$ is the nonlinear part. Also, the *response system* is described by the chaotic dynamics

$$\dot{y} = By + g(y) + u \tag{2}$$

where $B$ is the $n \times n$ matrix of the system parameters, $g : R^n \to R^n$ is the nonlinear part and $u \in R^n$ is the active controller to be designed.

For the pair of chaotic systems (1) and (2), the design goal of the anti-synchronization problem is to construct a feedback controller $u$, which *anti-synchronizes* their states for all $x(0), y(0) \in R^n$. The *anti-synchronization error* is defined as

$$e = y + x, \tag{3}$$

The error dynamics is obtained as

$$\dot{e} = By + Ax + g(y) + f(x) + u \tag{4}$$

The design goal is to find a feedback controller $u$ so that

$$\lim_{t \to \infty} \|e(t)\| = 0 \text{ for all } e(0) \in R^{\boldsymbol{n}} \tag{5}$$

Using the matrix method, we consider a candidate Lyapunov function

$$V(e) = e^T P e, \tag{6}$$

where $P$ is a positive definite matrix.
It is noted that $V : R^n \to R$ is a positive definite function.
If we find a feedback controller $u$ so that

$$\dot{V}(e) = -e^T Q e, \tag{7}$$

where $Q$ is a positive definite matrix, then $\dot{V} : R^n \to R$ is a negative definite function.

Thus, by Lyapunov stability theory [26], the error dynamics (4) is globally exponentially stable. When the system parameters in (1) and (2) are unknown, we apply adaptive control theory to construct a parameter update law for determining the estimates of the unknown parameters.

## 3. HYPERCHAOTIC WANG AND HYPERCHAOTIC LI SYSTEMS

The hyperchaotic Wang system ([23], 2008) is given by

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1) + x_2 x_3 \\ \dot{x}_2 &= c x_1 - x_1 x_3 - x_2 - 0.5 x_4 \\ \dot{x}_3 &= -d x_3 + x_1 x_2 \\ \dot{x}_4 &= b x_4 + 0.5 x_1 x_3\end{aligned} \tag{8}$$

where $a, b, c, d$ are constant, positive parameters of the system.

The Wang system (8) depicts a hyperchaotic attractor for the parametric values

$$a = 40, \quad b = 1.7, \quad c = 88, \quad d = 3 \tag{9}$$

The Lyapunov exponents of the system (8) are determined as

$$\lambda_1 = 3.2553, \quad \lambda_2 = 1.4252, \quad \lambda_3 = 0, \quad \lambda_4 = -46.9794 \tag{10}$$

Since there are two positive Lyapunov exponents in (10), the Wang system (8) is hyperchaotic for the parametric values (9).

Figure 1 shows the phase portrait of the hyperchaotic Wang system.
The hyperchaotic Li system ([24], 2005) is given by

$$\begin{aligned}\dot{x}_1 &= \alpha(x_2 - x_1) + x_4 \\ \dot{x}_2 &= \delta x_1 - x_1 x_3 + \gamma x_2 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2 \\ \dot{x}_4 &= x_2 x_3 + r x_4\end{aligned} \tag{11}$$

where $\alpha, \beta, \gamma, \delta, r$ are constant, positive parameters of the system.

The Li system (11) depicts a hyperchaotic attractor for the parametric values

$$\alpha = 35, \quad \beta = 3, \quad \gamma = 12, \quad \delta = 7, \quad r = 0.58 \tag{12}$$

The Lyapunov exponents of the system (11) for the parametric values in (12) are

$$\lambda_1 = 0.5011, \quad \lambda_2 = 0.1858, \quad \lambda_3 = 0, \quad \lambda_4 = -26.1010 \tag{13}$$

Since there are two positive Lyapunov exponents in (13), the Li system (11) is hyperchaotic for the parametric values (12). Figure 2 shows the phase portrait of the hyperchaotic Li system.

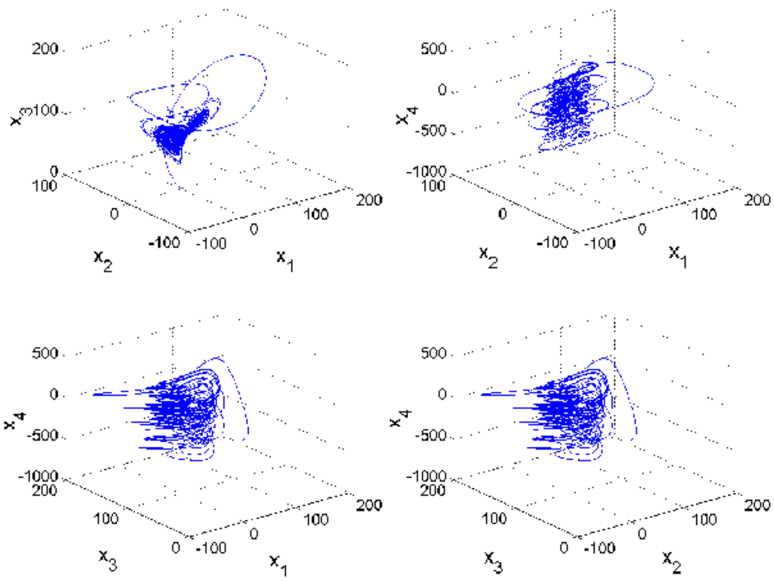


Figure 1. Hyperchaotic Attractor of the Hyperchaotic Wang System

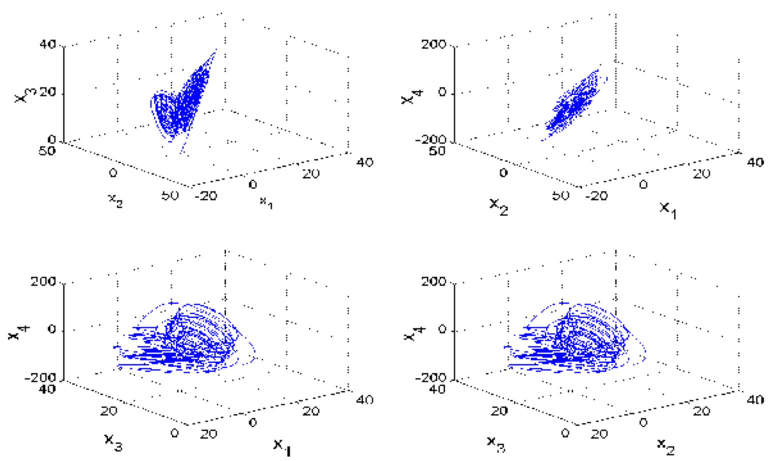


Figure 2. Hyperchaotic Attractor of the Hyperchaotic Li System

## 4. ANTI-SYNCHRONIZATION OF HYPERCHAOTIC WANG SYSTEMS VIA ADAPTIVE CONTROL

In this section, we derive new results for designing a controller for the anti-synchronization of identical hyperchaotic Wang systems (2008) with unknown parameters via adaptive control.
The drive system is the hyperchaotic Wang dynamics given by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_2 x_3 \\ \dot{x}_2 &= c x_1 - x_1 x_3 - x_2 - 0.5 x_4 \\ \dot{x}_3 &= -d x_3 + x_1 x_2 \\ \dot{x}_4 &= b x_4 + 0.5 x_1 x_3 \end{aligned} \tag{14}$$

where $a, b, c, d$ are unknown parameters of the system and $x \in R^4$ is the state.
The response system is the controlled hyperchaotic Wang dynamics given by

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + y_2 y_3 + u_1 \\ \dot{y}_2 &= c y_1 - y_1 y_3 - y_2 - 0.5 y_4 + u_2 \\ \dot{y}_3 &= -d y_3 + y_1 y_2 + u_3 \\ \dot{y}_4 &= b y_4 + 0.5 y_1 y_3 + u_4 \end{aligned} \tag{15}$$

where $y \in R^4$ is the state and $u_1, u_2, u_3, u_4$ are the adaptive controllers to be designed.
For the anti-synchronization, the error $e$ is defined as

$$e_1 = y_1 + x_1, \quad e_2 = y_2 + x_2, \quad e_3 = y_3 + x_3, \quad e_4 = y_4 + x_4 \tag{16}$$

Then we derive the error dynamics as

$$\begin{aligned} \dot{e}_1 &= a(e_2 - e_1) + y_2 y_3 + x_2 x_3 + u_1 \\ \dot{e}_2 &= c e_1 - e_2 - 0.5 e_4 - y_1 y_3 - x_1 x_3 + u_2 \\ \dot{e}_3 &= -d e_3 + y_1 y_2 + x_1 x_2 + u_3 \\ \dot{e}_4 &= b e_4 + 0.5(y_1 y_3 + x_1 x_3) + u_4 \end{aligned} \tag{17}$$

The adaptive controller to solve the anti-synchronization problem is taken as

$$\begin{aligned} u_1 &= -\hat{a}(t)(e_2 - e_1) - y_2 y_3 - x_2 x_3 - k_1 e_1 \\ u_2 &= -\hat{c}(t) e_1 + e_2 + 0.5 e_4 + y_1 y_3 + x_1 x_3 - k_2 e_2 \\ u_3 &= \hat{d}(t) e_3 - y_1 y_2 - x_1 x_2 - k_3 e_3 \\ u_4 &= -\hat{b}(t) e_4 - 0.5(y_1 y_3 + x_1 x_3) - k_4 e_4 \end{aligned} \tag{18}$$

In Eq. (18), $k_i$, $(i = 1, 2, 3, 4)$ are positive gains and $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t)$ are estimates for the unknown parameters $a, b, c, d$ respectively.

By the substitution of (18) into (17), the error dynamics is obtained as

$$\begin{aligned} \dot{e}_1 &= (a - \hat{a}(t))(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= (c - \hat{c}(t)) e_1 - k_2 e_2 \\ \dot{e}_3 &= -(d - \hat{d}(t)) e_3 - k_3 e_3 \\ \dot{e}_4 &= (b - \hat{b}(t)) e_4 - k_4 e_4 \end{aligned} \tag{19}$$

Next, we define the parameter estimation errors as

$$e_a(t) = a - \hat{a}(t),\ e_b(t) = b - \hat{b}(t),\ e_c(t) = c - \hat{c}(t),\ e_d(t) = d - \hat{d}(t) \tag{20}$$

Upon differentiation, we get

$$\dot{e}_a(t) = -\dot{\hat{a}}(t),\ \dot{e}_b(t) = -\dot{\hat{b}}(t),\ \dot{e}_c(t) = -\dot{\hat{c}}(t),\ \dot{e}_d(t) = -\dot{\hat{d}}(t) \tag{21}$$

Substituting (20) into the error dynamics (19), we obtain

$$\begin{aligned} \dot{e}_1 &= e_a(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= e_c e_1 - k_2 e_2 \\ \dot{e}_3 &= -e_d e_3 - k_3 e_3 \\ \dot{e}_4 &= e_b e_4 - k_4 e_4 \end{aligned} \tag{22}$$

We consider the candidate Lyapunov function

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2\right) \tag{23}$$

Differentiating (23) along the dynamics (21) and (22), we obtain

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a\left[e_1(e_2 - e_1) - \dot{\hat{a}}\right] + e_b\left(e_4^2 - \dot{\hat{b}}\right) \\ &+ e_c\left(e_1 e_2 - \dot{\hat{c}}\right) + e_d\left(-e_3^2 - \dot{\hat{d}}\right) \end{aligned} \tag{24}$$

In view of (24), we choose the following parameter update law:

$$\begin{aligned} \dot{\hat{a}} &= e_1(e_2 - e_1) + k_5 e_a \\ \dot{\hat{b}} &= e_4^2 + k_6 e_b \\ \dot{\hat{c}} &= e_1 e_2 + k_7 e_c \\ \dot{\hat{d}} &= -e_3^2 + k_8 e_d \end{aligned} \tag{25}$$

Next, we prove the following main result of this section.

**Theorem 4.1** The adaptive control law defined by Eq. (18) along with the parameter update law defined by Eq. (25), where $k_i, (i = 1, 2, \ldots, 8)$ are positive constants, render global and exponential anti-synchronization of the identical hyperchaotic Wang systems (14) and (15) with unknown parameters for all initial conditions $x(0), y(0) \in R^4$. In addition, the parameter estimation errors $e_a(t), e_b(t), e_c(t), e_d(t)$ globally and exponentially converge to zero for all initial conditions.

**Proof.** The proof is via Lyapunov stability theory [25] by taking $V$ defined by Eq. (23) as the candidate Lyapunov function. Substituting the parameter update law (25) into (24), we get

$$\dot{V}(e) = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 - k_5e_a^2 - k_6e_b^2 - k_7e_c^2 - k_8e_d^2 \tag{26}$$

which is a negative definite function on $R^8$. This completes the proof. ■

Next, we demonstrate our adaptive anti-synchronization results with MATLAB simulations. The classical fourth order R-K method with time-step $h = 10^{-8}$ has been used to solve the hyperchaotic Wang systems (14) and (15) with the adaptive controller defined by (18) and parameter update law defined by (25).

The feedback gains in the adaptive controller (18) are taken as $k_i = 5, \ (i = 1,\ldots,8).$
The parameters of the hyperchaotic Wang systems are taken as in the hyperchaotic case, *i.e.*

$$a = 40, \quad b = 1.7, \quad c = 88, \quad d = 3$$

For simulations, the initial conditions of the drive system (14) are taken as

$$x_1(0) = 37, \quad x_2(0) = -16, \quad x_3(0) = 14, \quad x_4(0) = 11$$

Also, the initial conditions of the response system (15) are taken as

$$y_1(0) = 21, \quad y_2(0) = 32, \quad y_3(0) = -28, \quad y_4(0) = -8$$

Also, the initial conditions of the parameter estimates are taken as

$$\hat{a}(0) = 12, \quad \hat{b}(0) = 4, \quad \hat{c}(0) = -6, \quad \hat{d}(0) = 5$$

Figure 3 depicts the anti-synchronization of the identical hyperchaotic Wang systems.
Figure 4 depicts the time-history of the anti-synchronization errors $e_1, e_2, e_3, e_4$.
Figure 5 depicts the time-history of the parameter estimation errors $e_a, e_b, e_c, e_d$.

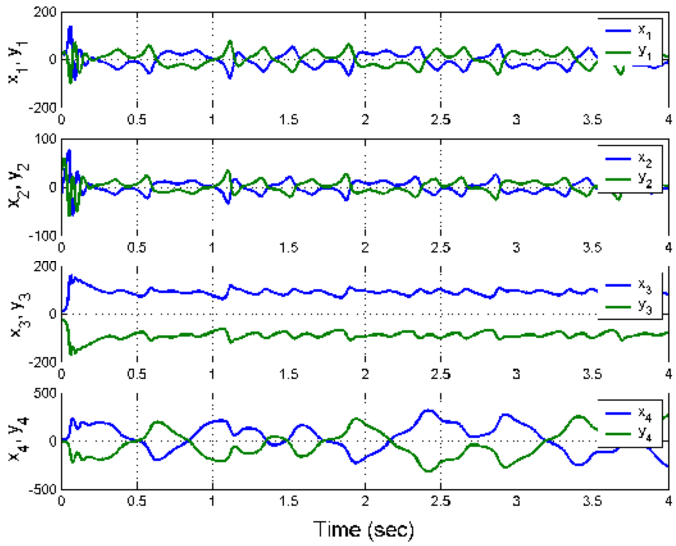


Figure 3. Anti-Synchronization of Identical Hyperchaotic Wang Systems

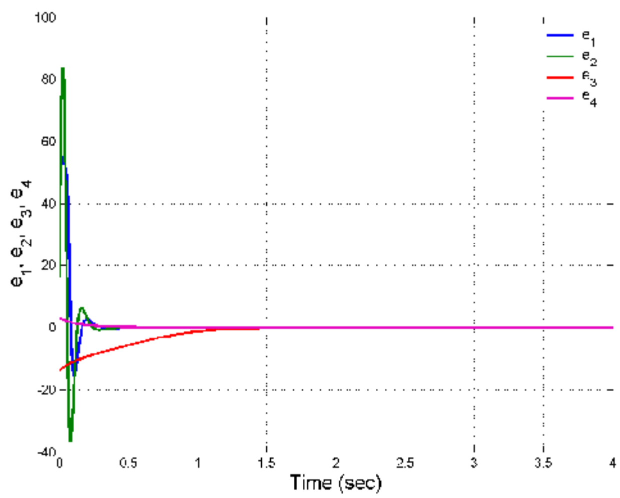


Figure 4. Time-History of the Anti-Synchronization Errors $e_1, e_2, e_3, e_4$

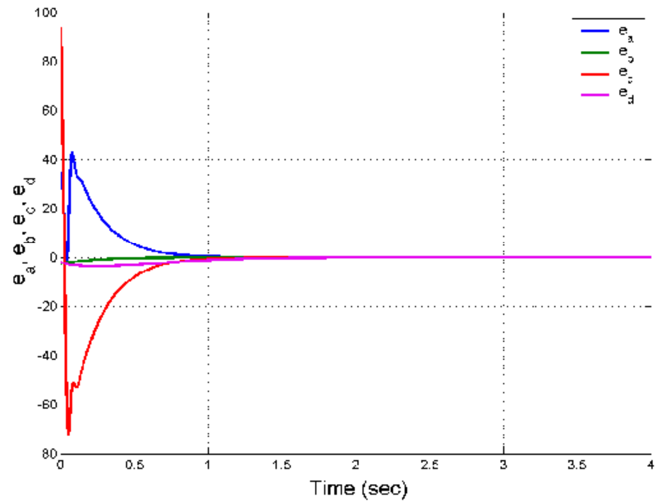


Figure 5. Time-History of the Parameter Estimation Errors $e_a, e_b, e_c, e_d$

## 5. ANTI-SYNCHRONIZATION OF HYPERCHAOTIC LI SYSTEMS VIA ADAPTIVE CONTROL

In this section, we derive new results for designing a controller for the anti-synchronization of identical hyperchaotic Li systems (2005) with unknown parameters via adaptive control.
The drive system is the hyperchaotic Li dynamics given by

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - x_1) + x_4 \\ \dot{x}_2 &= \delta x_1 - x_1 x_3 + \gamma x_2 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2 \\ \dot{x}_4 &= x_2 x_3 + r x_4 \end{aligned} \qquad (27)$$

where $\alpha, \beta, \gamma, \delta, r$ are unknown parameters of the system and $x \in R^4$ is the state.
The response system is the controlled hyperchaotic Li dynamics given by

$$\begin{aligned} \dot{y}_1 &= \alpha(y_2 - y_1) + y_4 + u_1 \\ \dot{y}_2 &= \delta y_1 - y_1 y_3 + \gamma y_2 + u_2 \\ \dot{y}_3 &= -\beta y_3 + y_1 y_2 + u_3 \\ \dot{y}_4 &= y_2 y_3 + r y_4 + u_4 \end{aligned} \qquad (28)$$

where $y \in R^4$ is the state and $u_1, u_2, u_3, u_4$ are the adaptive controllers to be designed.

For the anti-synchronization, the error $e$ is defined as

$$e_1 = y_1 + x_1, \quad e_2 = y_2 + x_2, \quad e_3 = y_3 + x_3, \quad e_4 = y_4 + x_4 \qquad (29)$$

Then we derive the error dynamics as

$$\begin{aligned}\dot{e}_1 &= \alpha(e_2 - e_1) + e_4 + u_1 \\ \dot{e}_2 &= \delta e_1 + \gamma e_2 - y_1 y_3 - x_1 x_3 + u_2 \\ \dot{e}_3 &= -\beta e_3 + y_1 y_2 + x_1 x_2 + u_3 \\ \dot{e}_4 &= r e_4 + y_2 y_3 + x_2 x_3 + u_4\end{aligned} \tag{30}$$

The adaptive controller to solve the anti-synchronization problem is taken as

$$\begin{aligned}u_1 &= -\hat{\alpha}(t)(e_2 - e_1) - e_4 - k_1 e_1 \\ u_2 &= -\hat{\delta}(t) e_1 - \hat{\gamma}(t) e_2 + y_1 y_3 + x_1 x_3 - k_2 e_2 \\ u_3 &= \hat{\beta}(t) e_3 - y_1 y_2 - x_1 x_2 - k_3 e_3 \\ u_4 &= -\hat{r}(t) e_4 - y_2 y_3 - x_2 x_3 - k_4 e_4\end{aligned} \tag{31}$$

In Eq. (31), $k_i$, $(i = 1, 2, 3, 4)$ are positive gains and $\hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t), \hat{\delta}(t), \hat{r}(t)$ are estimates for the unknown parameters $\alpha, \beta, \gamma, \delta, r$ respectively.

By the substitution of (31) into (30), the error dynamics is obtained as

$$\begin{aligned}\dot{e}_1 &= (\alpha - \hat{\alpha}(t))(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= (\delta - \hat{\delta}(t)) e_1 + (\gamma - \hat{\gamma}(t)) e_2 - k_2 e_2 \\ \dot{e}_3 &= -(\beta - \hat{\beta}(t)) e_3 - k_3 e_3 \\ \dot{e}_4 &= (r - \hat{r}(t)) e_4 - k_4 e_4\end{aligned} \tag{32}$$

Next, we define the parameter estimation errors as

$$e_\alpha(t) = \alpha - \hat{\alpha}(t), e_\beta(t) = \beta - \hat{\beta}(t), e_\gamma(t) = \gamma - \hat{\gamma}(t), e_\delta(t) = \delta - \hat{\delta}(t), e_r(t) = r - \hat{r}(t) \tag{33}$$

Upon differentiation, we get

$$\dot{e}_\alpha(t) = -\dot{\hat{\alpha}}(t),\ \dot{e}_\beta(t) = -\dot{\hat{\beta}}(t),\ \dot{e}_\gamma(t) = -\dot{\hat{\gamma}}(t),\ \dot{e}_\delta(t) = -\dot{\hat{\delta}}(t), \dot{e}_r(t) = -\dot{\hat{r}}(t) \tag{34}$$

Substituting (33) into the error dynamics (32), we obtain

$$\begin{aligned}\dot{e}_1 &= e_\alpha (e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= e_\delta e_1 + e_\gamma e_2 - k_2 e_2 \\ \dot{e}_3 &= -e_\beta e_3 - k_3 e_3 \\ \dot{e}_4 &= e_r e_4 - k_4 e_4\end{aligned} \tag{35}$$

We consider the candidate Lyapunov function

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2 + e_r^2\right) \tag{36}$$

Differentiating (36) along the dynamics (34) and (35), we obtain

$$\begin{aligned} \dot{V} = & -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_\alpha\left[e_1(e_2 - e_1) - \dot{\hat{\alpha}}\right] + e_\beta\left(-e_3^2 - \dot{\hat{\beta}}\right) \\ & + e_\gamma\left(e_2^2 - \dot{\hat{\gamma}}\right) + e_\delta\left(e_1 e_2 - \dot{\hat{\delta}}\right) + e_r\left(e_4^2 - \dot{\hat{r}}\right) \end{aligned} \tag{37}$$

In view of (37), we choose the following parameter update law:

$$\begin{aligned} \dot{\hat{\alpha}} &= e_1(e_2 - e_1) + k_5 e_a \\ \dot{\hat{\beta}} &= -e_3^2 + k_6 e_\beta \\ \dot{\hat{\gamma}} &= e_2^2 + k_7 e_\gamma \\ \dot{\hat{\delta}} &= e_1 e_2 + k_8 e_\delta \\ \dot{\hat{r}} &= e_4^2 + k_9 e_r \end{aligned} \tag{38}$$

Next, we prove the following main result of this section.

**Theorem 5.1** The adaptive control law defined by Eq. (31) along with the parameter update law defined by Eq. (38), where $k_i, (i = 1, 2, \ldots, 9)$ are positive constants, render global and exponential anti-synchronization of the identical hyperchaotic Li systems (27) and (28) with unknown parameters for all initial conditions $x(0), y(0) \in R^4$. In addition, the parameter estimation errors $e_\alpha(t), e_\beta(t), e_\gamma(t), e_\delta(t), e_r(t)$ globally and exponentially converge to zero for all initial conditions.

**Proof.** The proof is via Lyapunov stability theory [25] by taking $V$ defined by Eq. (36) as the candidate Lyapunov function. Substituting the parameter update law (38) into (37), we get

$$\dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_\alpha^2 - k_6 e_\beta^2 - k_7 e_\gamma^2 - k_8 e_\delta^2 - k_9 e_r^2 \tag{39}$$

which is a negative definite function on $R^9$. This completes the proof. ■

Next, we demonstrate our adaptive anti-synchronization results with MATLAB simulations. The classical fourth order R-K method with time-step $h = 10^{-8}$ has been used to solve the hyperchaotic Li systems (27) and (28) with the adaptive controller defined by (31) and parameter update law defined by (38). The feedback gains in the adaptive controller (31) are taken as $k_i = 5, \ (i = 1, \ldots, 9)$.

The parameters of the hyperchaotic Li systems are taken as in the hyperchaotic case, *i.e.*

$$\alpha = 35, \quad \beta = 3, \quad \gamma = 12, \quad \delta = 7, \quad r = 0.58$$

For simulations, the initial conditions of the drive system (27) are taken as

$$x_1(0)=7,\ \ x_2(0)=-26,\ \ x_3(0)=12,\ \ x_4(0)=14$$

Also, the initial conditions of the response system (28) are taken as

$$y_1(0)=-21,\ \ y_2(0)=28,\ \ y_3(0)=-18,\ \ y_4(0)=29$$

Also, the initial conditions of the parameter estimates are taken as

$$\hat{\alpha}(0)=7,\ \ \hat{\beta}(0)=15,\ \ \hat{\gamma}(0)=5,\ \ \hat{\delta}(0)=4,\ \ \hat{r}(0)=-3$$

Figure 6 depicts the anti-synchronization of the identical hyperchaotic Li systems.
Figure 7 depicts the time-history of the anti-synchronization errors $e_1, e_2, e_3, e_4$.
Figure 8 depicts the time-history of the parameter estimation errors $e_\alpha, e_\beta, e_\gamma, e_\delta, e_r$.

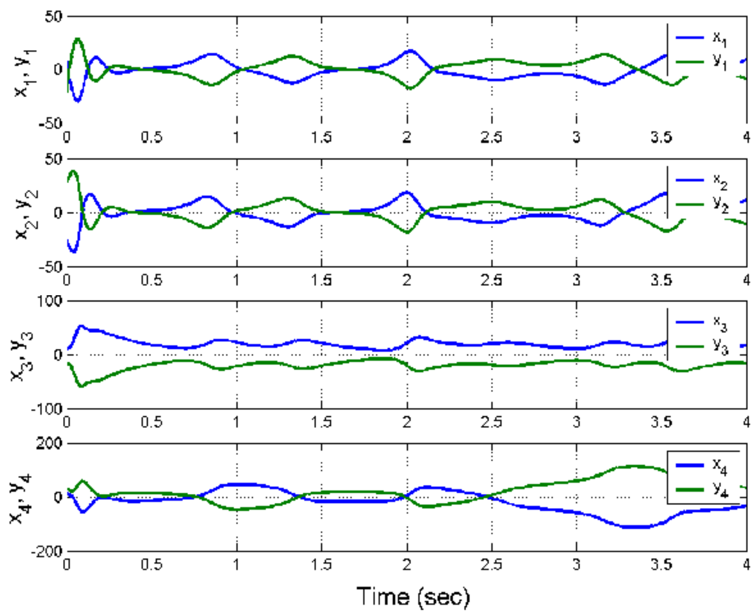


Figure 6. Anti-Synchronization of Identical Hyperchaotic Li Systems

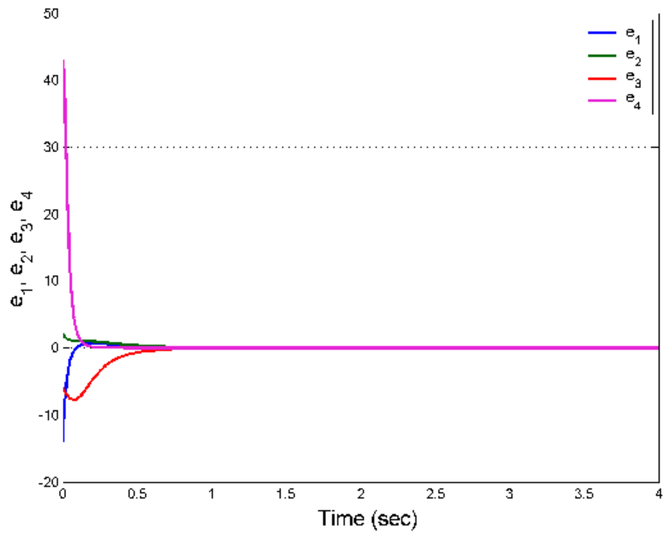


Figure 7. Time-History of the Anti-Synchronization Errors $e_1, e_2, e_3, e_4$

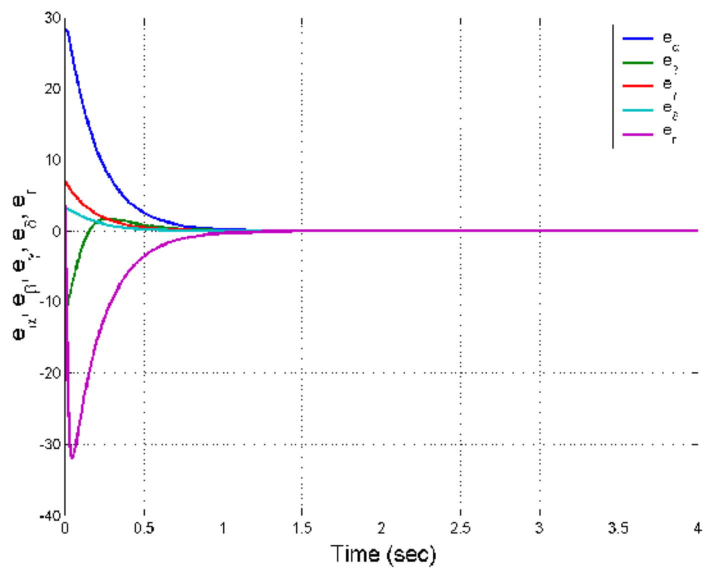


Figure 8. Time-History of the Parameter Estimation Errors $e_\alpha, e_\beta, e_\gamma, e_\delta, e_r$

## 6.ANTI-SYNCHRONIZATION OF HYPERCHAOTIC WANG AND HYPERCHAOTIC LI SYSTEMS VIA ADAPTIVE CONTROL

In this section, we derive new results for designing a controller for the anti-synchronization of non-identical hyperchaotic Wang system (2009) and hyperchaotic Li system (2005) with unknown parameters via adaptive control.

The drive system is the hyperchaotic Wang dynamics given by

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_2 x_3 \\
\dot{x}_2 &= cx_1 - x_1 x_3 - x_2 - 0.5x_4 \\
\dot{x}_3 &= -dx_3 + x_1 x_2 \\
\dot{x}_4 &= bx_4 + 0.5x_1 x_3
\end{aligned} \tag{40}$$

where $a, b, c, d$ are unknown parameters of the system and $x \in R^4$ is the state.
The response system is the controlled hyperchaotic Li dynamics given by

$$\begin{aligned}
\dot{y}_1 &= \alpha(y_2 - y_1) + y_4 + u_1 \\
\dot{y}_2 &= \delta y_1 - y_1 y_3 + \gamma y_2 + u_2 \\
\dot{y}_3 &= -\beta y_3 + y_1 y_2 + u_3 \\
\dot{y}_4 &= y_2 y_3 + r y_4 + u_4
\end{aligned} \tag{41}$$

where $\alpha, \beta, \gamma, \delta, r$ are unknown parameters, $y \in R^4$ is the state and $u_1, u_2, u_3, u_4$ are the adaptive controllers to be designed.
For the anti-synchronization, the error $e$ is defined as

$$e_1 = y_1 + x_1, \quad e_2 = y_2 + x_2, \quad e_3 = y_3 + x_3, \quad e_4 = y_4 + x_4 \tag{42}$$

Then we derive the error dynamics as

$$\begin{aligned}
\dot{e}_1 &= a(x_2 - x_1) + \alpha(y_2 - y_1) + y_4 + x_2 x_3 + u_1 \\
\dot{e}_2 &= cx_1 + \delta y_1 - x_2 + \gamma y_2 - 0.5x_4 - x_1 x_3 - y_1 y_3 + u_2 \\
\dot{e}_3 &= -dx_3 - \beta y_3 + y_1 y_2 + x_1 x_2 + u_3 \\
\dot{e}_4 &= bx_4 + ry_4 + 0.5x_1 x_3 + y_2 y_3 + u_4
\end{aligned} \tag{43}$$

The adaptive controller to solve the anti-synchronization problem is taken as

$$\begin{aligned}
u_1 &= -\hat{a}(t)(x_2 - x_1) - \hat{\alpha}(t)(y_2 - y_1) - y_4 - x_2 x_3 - k_1 e_1 \\
u_2 &= -\hat{c}(t)x_1 - \hat{\delta}(t)y_1 + x_2 - \hat{\gamma}(t)y_2 + 0.5x_4 + x_1 x_3 + y_1 y_3 - k_2 e_2 \\
u_3 &= \hat{d}(t)x_3 + \hat{\beta}(t)y_3 - y_1 y_2 - x_1 x_2 - k_3 e_3 \\
u_4 &= -\hat{b}(t)x_4 - \hat{r}(t)y_4 - 0.5x_1 x_3 - y_2 y_3 - k_4 e_4
\end{aligned} \tag{44}$$

In Eq. (44), $k_i, \ (i = 1, 2, 3, 4)$ are positive gains and $\hat{a}(t), \ \hat{b}(t), \ \hat{c}(t), \ \hat{d}(t), \ \hat{\alpha}(t), \ \hat{\beta}(t), \ \hat{\gamma}(t), \ \hat{\delta}(t), \ \hat{r}(t)$ are estimates for the unknown parameters $a, b, c, d, \alpha, \beta, \gamma, \delta, r$ respectively.
By the substitution of (44) into (43), the error dynamics is obtained as

$$\begin{aligned}
\dot{e}_1 &= (a - \hat{a}(t))(x_2 - x_1) + (\alpha - \hat{\alpha}(t))(y_2 - y_1) - k_1 e_1 \\
\dot{e}_2 &= (c - \hat{c}(t))x_1 + (\delta - \hat{\delta}(t))y_1 + (\gamma - \hat{\gamma}(t))y_2 - k_2 e_2 \\
\dot{e}_3 &= -(d - \hat{d}(t))x_3 - (\beta - \hat{\beta}(t))y_3 - k_3 e_3 \\
\dot{e}_4 &= (b - \hat{b}(t))x_4 + (r - \hat{r}(t))y_4 - k_4 e_4
\end{aligned} \tag{45}$$

Next, we define the parameter estimation errors as

$$\begin{aligned} & e_a(t) = a - \hat{a}(t),\ e_b(t) = b - \hat{b}(t),\ e_c(t) = c - \hat{c}(t),\ e_d(t) = d - \hat{d}(t) \\ & e_\alpha(t) = \alpha - \hat{\alpha}(t),\ e_\beta(t) = \beta - \hat{\beta}(t),\ e_\gamma(t) = \gamma - \hat{\gamma}(t),\ e_\delta(t) = \delta - \hat{\delta}(t), e_r(t) = r - \hat{r}(t) \end{aligned} \tag{46}$$

Upon differentiation, we get

$$\begin{aligned} & \dot{e}_a(t) = -\dot{\hat{a}}(t),\ \dot{e}_b(t) = -\dot{\hat{b}}(t),\ \dot{e}_c(t) = -\dot{\hat{c}}(t),\ \dot{e}_d(t) = -\dot{\hat{d}}(t) \\ & \dot{e}_\alpha(t) = -\dot{\hat{\alpha}}(t),\ \dot{e}_\beta(t) = -\dot{\hat{\beta}}(t),\ \dot{e}_\gamma(t) = -\dot{\hat{\gamma}}(t),\ \dot{e}_\delta(t) = -\dot{\hat{\delta}}(t), \dot{e}_r(t) = -\dot{\hat{r}}(t) \end{aligned} \tag{47}$$

Substituting (46) into the error dynamics (45), we obtain

$$\begin{aligned} \dot{e}_1 &= e_a(x_2 - x_1) + e_\alpha(y_2 - y_1) - k_1 e_1 \\ \dot{e}_2 &= e_c x_1 + e_\delta y_1 + e_\gamma y_2 - k_2 e_2 \\ \dot{e}_3 &= -e_d x_3 - e_\beta y_3 - k_3 e_3 \\ \dot{e}_4 &= e_b x_4 + e_r y_4 - k_4 e_4 \end{aligned} \tag{48}$$

We consider the candidate Lyapunov function

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2 + e_r^2\right) \tag{49}$$

Differentiating (49) along the dynamics (47) and (48), we obtain

$$\begin{aligned} \dot{V} = & -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a\left[e_1(x_2 - x_1) - \dot{\hat{a}}\right] + e_b\left(e_4 x_4 - \dot{\hat{b}}\right) + e_c\left(e_2 x_1 - \dot{\hat{c}}\right) \\ & + e_d\left(-e_3 x_3 - \dot{\hat{d}}\right) + e_\alpha\left[e_1(y_2 - y_1) - \dot{\hat{\alpha}}\right] + e_\beta\left(-e_3 y_3 - \dot{\hat{\beta}}\right) \\ & + e_\gamma\left(e_2 y_2 - \dot{\hat{\gamma}}\right) + e_\delta\left(e_2 y_1 - \dot{\hat{\delta}}\right) + e_r\left(e_4 y_4 - \dot{\hat{r}}\right) \end{aligned} \tag{50}$$

In view of (50), we choose the following parameter update law:

$$\begin{aligned} & \dot{\hat{a}} = e_1(x_2 - x_1) + k_5 e_a, && \dot{\hat{\alpha}} = e_1(y_2 - y_1) + k_9 e_a \\ & \dot{\hat{b}} = e_4 x_4 + k_6 e_b, && \dot{\hat{\beta}} = -e_3 y_3 + k_{10} e_\beta \\ & \dot{\hat{c}} = e_2 x_1 + k_7 e_c, && \dot{\hat{\gamma}} = e_2 y_2 + k_{11} e_\gamma \\ & \dot{\hat{d}} = -e_3 x_3 + k_8 e_d, && \dot{\hat{\delta}} = e_2 y_1 + k_{12} e_\delta \\ & \dot{\hat{r}} = e_4 y_4 + k_{13} e_r \end{aligned} \tag{51}$$

Next, we prove the following main result of this section.

**Theorem 6.1** The adaptive control law defined by Eq. (44) along with the parameter update law defined by Eq. (51), where $k_i, (i = 1, 2, \ldots, 13)$ are positive constants, render global and exponential anti-synchronization of the non-identical hyperchaotic Wang system (40) and hyperchaotic Li system (41) with unknown parameters for all initial conditions $x(0), y(0) \in R^4$. In addition, all the parameter estimation errors globally and exponentially converge to zero for all initial conditions.

**Proof.** The proof is via Lyapunov stability theory [25] by taking $V$ defined by Eq. (49) as the candidate Lyapunov function. Substituting the parameter update law (51) into (50), we get

$$\begin{aligned}\dot{V}(e) = & -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_a^2 - k_6 e_b^2 - k_7 e_c^2 - k_8 e_d^2 \\ & - k_9 e_\alpha^2 - k_{10} e_\beta^2 - k_{11} e_\gamma^2 - k_{12} e_\delta^2 - k_{13} e_r^2\end{aligned} \quad (52)$$

which is a negative definite function on $R^{13}$. This completes the proof. ■

Next, we demonstrate our adaptive anti-synchronization results with MATLAB simulations. The classical fourth order R-K method with time-step $h = 10^{-8}$ has been used to solve the hyperchaotic systems (40) and (41) with the adaptive controller defined by (44) and parameter update law defined by (51).

The feedback gains in the adaptive controller (44) are taken as $k_i = 5, \ (i = 1, \ldots, 13)$.
The parameters of the hyperchaotic Wang and Li systems are taken as in the hyperchaotic case, *i.e.*

$$a = 40, \quad b = 1.7, \quad c = 88, \quad d = 3, \quad \alpha = 35, \quad \beta = 3, \quad \gamma = 12, \quad \delta = 7, \quad r = 0.58$$

For simulations, the initial conditions of the drive system (40) are taken as

$$x_1(0) = 12, \quad x_2(0) = -34, \quad x_3(0) = 31, \quad x_4(0) = 14$$

Also, the initial conditions of the response system (41) are taken as

$$y_1(0) = -25, \quad y_2(0) = 18, \quad y_3(0) = -12, \quad y_4(0) = 29$$

Also, the initial conditions of the parameter estimates are taken as

$$\begin{aligned}&\hat{a}(0) = 21, \quad \hat{b}(0) = 14, \quad \hat{c}(0) = -26, \quad \hat{d}(0) = 16 \\ &\hat{\alpha}(0) = 17, \quad \hat{\beta}(0) = -22 \quad \hat{\gamma}(0) = 15, \quad \hat{\delta}(0) = 11, \quad \hat{r}(0) = -7\end{aligned}$$

Figure 9 depicts the anti-synchronization of the hyperchaotic Wang and hyperchaotic Li systems.
Figure 10 depicts the time-history of the anti-synchronization errors $e_1, e_2, e_3, e_4$.
Figure 11 depicts the time-history of the parameter estimation errors $e_a, e_b, e_c, e_d$.
Figure 12 depicts the time-history of the parameter estimation errors $e_\alpha, e_\beta, e_\gamma, e_\delta, e_r$.

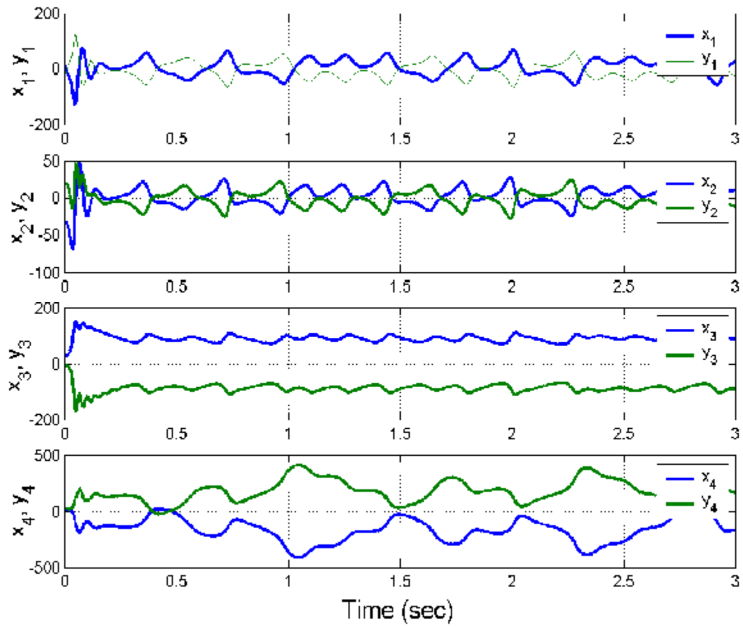


Figure 9. Anti-Synchronization of Hyperchaotic Wang and Hyperchaotic Li Systems

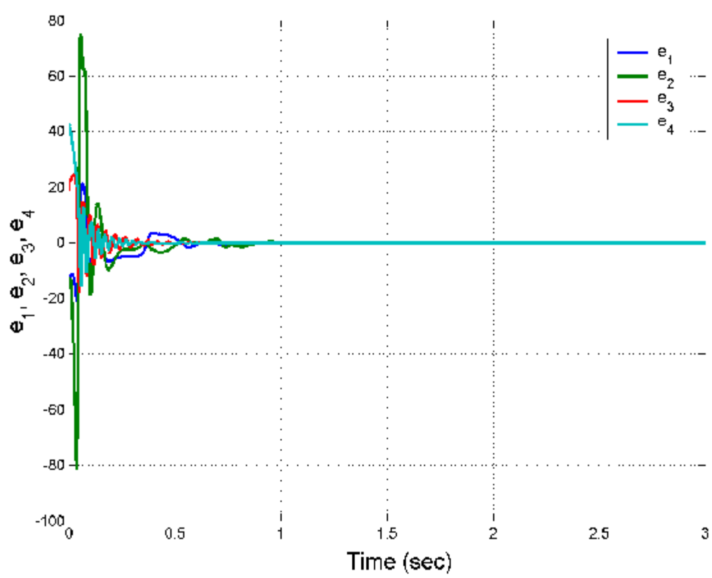


Figure 10. Time-History of the Anti-Synchronization Errors $e_1, e_2, e_3, e_4$

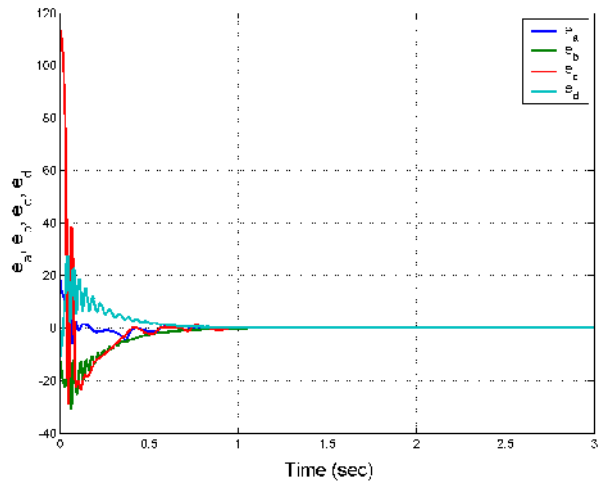


Figure 11. Time-History of the Parameter Estimation Errors $e_a, e_b, e_c, e_d$

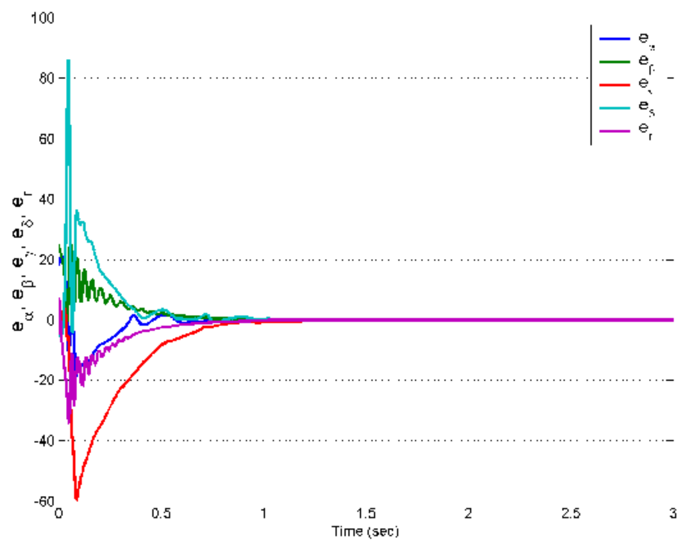


Figure 12. Time-History of the Parameter Estimation Errors $e_\alpha, e_\beta, e_\gamma, e_\delta, e_r$

## 7. CONCLUSIONS

This paper has used adaptive control theory and Lyapunov stability theory so as to solve the anti-synchronization problem for the anti-synchronization of hyperchaotic Wang system (2008) and hyperchaotic Li system (2005) with unknown parameters. Hyperchaotic systems are chaotic systems with two or more positive Lyapunov exponents and they have viable applications like chemical reactions, neural networks, secure communication, data encryption, neural networks, etc. MATLAB simulations were depicted to illustrate the various adaptive anti-synchronization results derived in this paper for the hyperchaotic Wang and Li systems.

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