ADAPTIVE CHAOS SYNCHRONIZATION OF UNCERTAIN HYPERCHAOTIC LORENZ AND HYPERCHAOTIC LÜ SYSTEMS

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ABSTRACT

In this paper, we apply adaptive control method to derive new results for the global chaos synchronization of identical hyperchaotic Lorenz systems (Jia, 2007), identical hyperchaotic Lü systems (Chen, et al., 2006) and non-identical hyperchaotic Lorenz and hyperchaotic Lü systems. In this paper, we shall assume that the parameters of both master and slave systems are unknown and we devise adaptive synchronizing schemes using the estimates of parameters for both master and slave systems. Our adaptive synchronization schemes derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is very effective and convenient to synchronize identical and non-identical hyperchaotic Lorenz and hyperchaotic Lü systems. Numerical simulations are shown to demonstrate the effectiveness of the proposed adaptive synchronization schemes for the uncertain hyperchaotic systems addressed in this paper.

KEYWORDS

Adaptive Control, Chaos Synchronization, Hyperchaotic Systems, Hyperchaotic Lorenz System; Hyperchaotic Lü System.

1. INTRODUCTION

Chaotic systems are dynamical systems that are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the *butterfly effect* [1].

Since the seminal work by Pecora and Carroll ([2], 1990), chaos synchronization problem has been studied extensively and intensively in the literature [2-29]. Chaos theory has been applied to a variety of fields such as physical systems [3], chemical systems [4], ecological systems [5], secure communications [6-8], etc.

In most of the chaos synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

In the last two decades, various schemes have been successfully applied for chaos synchronization such as PC method [2], OGY method [9], active control method [10-14], adaptive control method [15-18], time-delay feedback method [19], backstepping design method [20-22], sampled-data feedback method [23], sliding mode control method [24-27], etc.

In this paper, we apply adaptive control method to derive new results for the global chaos synchronization of identical hyperchaotic Lorenz systems ([28], 2007), identical hyperchaotic Lü systems ([29], 2006) and non-identical hyperchaotic Lorenz and hyperchaotic Lü systems. We assume that the parameters of the master and slave systems are unknown and we devise adaptive synchronizing schemes using the estimates of the parameters for both master and slave systems.

This paper has been organized as follows. In Section 2, we give a description of hyperchaotic Lorenz and hyperchaotic Lü systems. In Section 3, we discuss the adaptive synchronization of identical hyperchaotic Lorenz systems. In Section 4, we discuss the adaptive synchronization of identical hyperchaotic Lü systems. In Section 5, we discuss the adaptive synchronization of hyperchaotic Lorenz and hyperchaotic Lü systems. In Section 6, we summarize the main results obtained in this paper.

2. SYSTEMS DESCRIPTION

The hyperchaotic Lorenz system ([28], 2007) is described by

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1 x_3 + r x_1 - x_2 \\
\dot{x}_3 &= x_1 x_2 - b x_3 \\
\dot{x}_4 &= -x_1 x_3 + d x_4
\end{aligned} \tag{1}$$

where x_1, x_2, x_3, x_4 are the state variables and a, b, r, d are positive constant parameters of the system.

The system (1) is hyperchaotic when the parameter values are taken as

$$a = 10, \quad r = 28, \quad b = 8/3 \quad \text{and} \quad d = 1.3.$$

The state orbits of the hyperchaotic Lorenz system (1) are shown in Figure 1.

The hyperchaotic Lü system ([29], 2006) is described by

$$\begin{aligned}
\dot{x}_1 &= \alpha(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1 x_3 + \gamma x_2 \\
\dot{x}_3 &= x_1 x_2 - \beta x_3 \\
\dot{x}_4 &= x_1 x_3 + \delta x_4
\end{aligned} \tag{2}$$

where x_1, x_2, x_3, x_4 are the state variables and $\alpha, \beta, \gamma, \delta$ are positive constant parameters of the system.

The system (2) is hyperchaotic when the parameter values are taken as

$$\alpha = 36, \quad \beta = 3, \quad \gamma = 20 \quad \text{and} \quad \delta = 1.3.$$

The state orbits of the hyperchaotic Lü system (2) are shown in Figure 2.

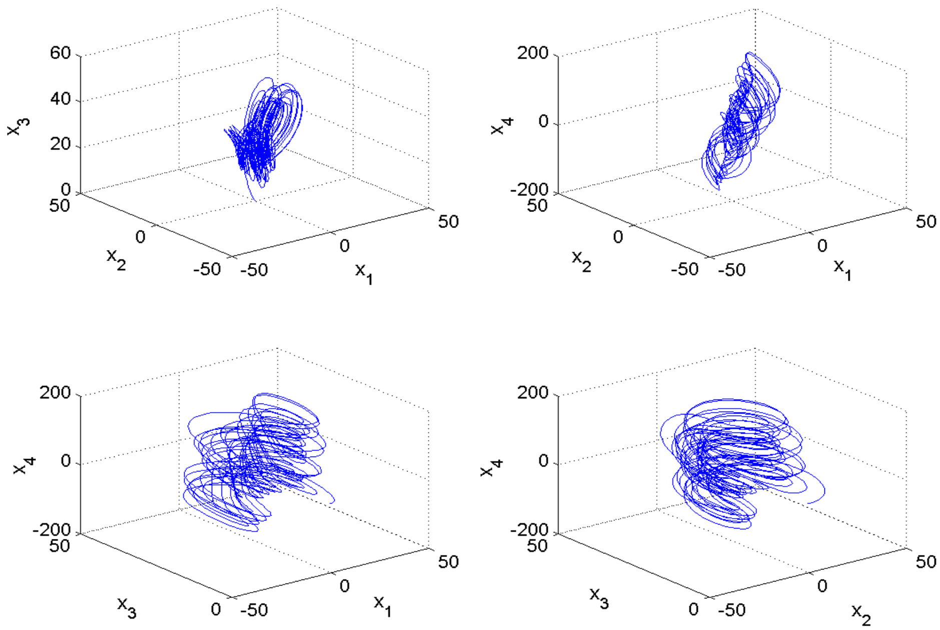

Figure 1. State Orbits of the Hyperchaotic Lorenz System

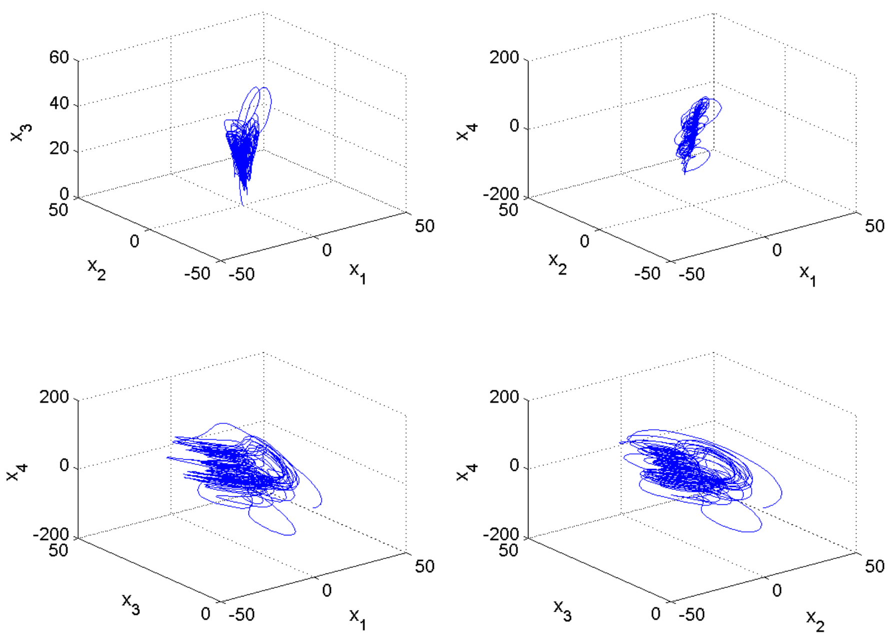

Figure 2. State Orbits of the Hyperchaotic Lü System

3. ADAPTIVE SYNCHRONIZATION OF IDENTICAL HYPERCHAOTIC LORENZ SYSTEMS

3.1 Theoretical Results

In this section, we discuss the adaptive synchronization of identical hyperchaotic Lorenz systems ([28], 2007), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Lorenz dynamics described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_4 \\ \dot{x}_2 &= -x_1 x_3 + r x_1 - x_2 \\ \dot{x}_3 &= x_1 x_2 - b x_3 \\ \dot{x}_4 &= -x_1 x_3 + d x_4 \end{aligned} \tag{3}$$

where x_1, x_2, x_3, x_4 are the states and a, b, r, d are unknown real constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Lorenz dynamics described by

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + y_4 + u_1 \\ \dot{y}_2 &= -y_1 y_3 + r y_1 - y_2 + u_2 \\ \dot{y}_3 &= y_1 y_2 - b y_3 + u_3 \\ \dot{y}_4 &= -y_1 y_3 + d y_4 + u_4 \end{aligned} \tag{4}$$

where y_1, y_2, y_3, y_4 are the states and u_1, u_2, u_3, u_4 are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

$$e_i = y_i - x_i, \quad (i = 1, 2, 3, 4) \tag{5}$$

The error dynamics is easily obtained as

$$\begin{aligned} \dot{e}_1 &= a(e_2 - e_1) + e_4 + u_1 \\ \dot{e}_2 &= r e_1 - e_2 - y_1 y_3 + x_1 x_3 + u_2 \\ \dot{e}_3 &= -b e_3 + y_1 y_2 - x_1 x_2 + u_3 \\ \dot{e}_4 &= d e_4 - y_1 y_3 + x_1 x_3 + u_4 \end{aligned} \tag{6}$$

Let us now define the adaptive control functions

$$\begin{aligned} u_1(t) &= -\hat{a}(e_2 - e_1) - e_4 - k_1 e_1 \\ u_2(t) &= -\hat{r} e_1 + e_2 + y_1 y_3 - x_1 x_3 - k_2 e_2 \\ u_3(t) &= \hat{b} e_3 - y_1 y_2 + x_1 x_2 - k_3 e_3 \\ u_4(t) &= -\hat{d} e_4 + y_1 y_3 - x_1 x_3 - k_4 e_4 \end{aligned} \tag{7}$$

where $\hat{a}, \hat{b}, \hat{r}$ and $\hat{d}$ are estimates of a, b, r and d, respectively, and $k_i, (i = 1, 2, 3, 4)$ are positive constants.

Substituting (7) into (6), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (a - \hat{a})(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= (r - \hat{r})e_1 - k_2 e_2 \\ \dot{e}_3 &= -(b - \hat{b})e_3 - k_3 e_3 \\ \dot{e}_4 &= (d - \hat{d})e_4 - k_4 e_4 \end{aligned} \tag{8}$$

Let us now define the parameter estimation errors as

$$e_a = a - \hat{a}, \quad e_b = b - \hat{b}, \quad e_r = r - \hat{r} \quad \text{and} \quad e_d = d - \hat{d}. \tag{9}$$

Substituting (9) into (8), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_a(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= e_r e_1 - k_2 e_2 \\ \dot{e}_3 &= -e_b e_3 - k_3 e_3 \\ \dot{e}_4 &= e_d e_4 - k_4 e_4 \end{aligned} \tag{10}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V(e_1, e_2, e_3, e_4, e_a, e_b, e_r, e_d) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_r^2 + e_d^2\right) \tag{11}$$

which is a positive definite function on R^8.

We also note that

$$\dot{e}_a = -\dot{\hat{a}}, \; \dot{e}_b = -\dot{\hat{b}}, \; \dot{e}_r = -\dot{\hat{r}} \quad \text{and} \quad \dot{e}_d = -\dot{\hat{d}} \tag{12}$$

Differentiating (11) along the trajectories of (10) and using (12), we obtain

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a\left[e_1(e_2 - e_1) - \dot{\hat{a}}\right] \\ &+ e_b\left[-e_3^2 - \dot{\hat{b}}\right] + e_r\left[e_1 e_2 - \dot{\hat{c}}\right] + e_d\left[e_4^2 - \dot{\hat{d}}\right] \end{aligned} \tag{13}$$

In view of Eq. (13), the estimated parameters are updated by the following law:

$$\begin{aligned} \dot{\hat{a}} &= e_1(e_2 - e_1) + k_5 e_a \\ \dot{\hat{b}} &= -e_3^2 + k_6 e_b \\ \dot{\hat{r}} &= e_1 e_2 + k_7 e_r \\ \dot{\hat{d}} &= e_4^2 + k_8 e_d \end{aligned} \tag{14}$$

where k_5, k_6, k_7 and k_8 are positive constants.

Substituting (14) into (12), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_a^2 - k_6 e_b^2 - k_7 e_r^2 - k_8 e_d^2 \tag{15}$$

which is a negative definite function on R^8.

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error $e_i, (i = 1,2,3,4)$ and the parameter estimation error e_a, e_b, e_r, e_d decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 1. *The identical hyperchaotic Lorenz systems (3) and (4) with unknown parameters are globally and exponentially synchronized by the adaptive control law (7), where the update law for the parameter estimates is given by (14) and* $k_i, (i = 1,2,\ldots,8)$ *are positive constants.* ■

3.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (3) and (4) with the adaptive control law (14) and the parameter update law (14) using MATLAB. We take $k_i = 2$ for $i = 1,2,\ldots,8$.

For the hyperchaotic Lorenz systems (3) and (4), the parameter values are taken as

$$a = 10, \quad r = 28, \quad b = 8/3 \quad \text{and} \quad d = 1.3.$$

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0) = 20, \quad \hat{b}(0) = 2, \quad \hat{r}(0) = 12, \quad \hat{d}(0) = 15.$$

The initial values of the master system (3) are taken as

$$x_1(0) = 8, \quad x_2(0) = 14, \quad x_3(0) = 36, \quad x_4(0) = 25.$$

The initial values of the slave system (4) are taken as

$$y_1(0) = 28, \quad y_2(0) = 6, \quad y_3(0) = 20, \quad y_4(0) = 12.$$

Figure 3 depicts the complete synchronization of the identical hyperchaotic Lorenz systems (3) and (4). Figure 4 shows that the estimated values of the parameters, viz. $\hat{a}, \hat{b}, \hat{r}$ and $\hat{d}$ converge to the system parameters $a = 10,\ \ r = 28,\ \ b = 8/3$ and $d = 1.3.$

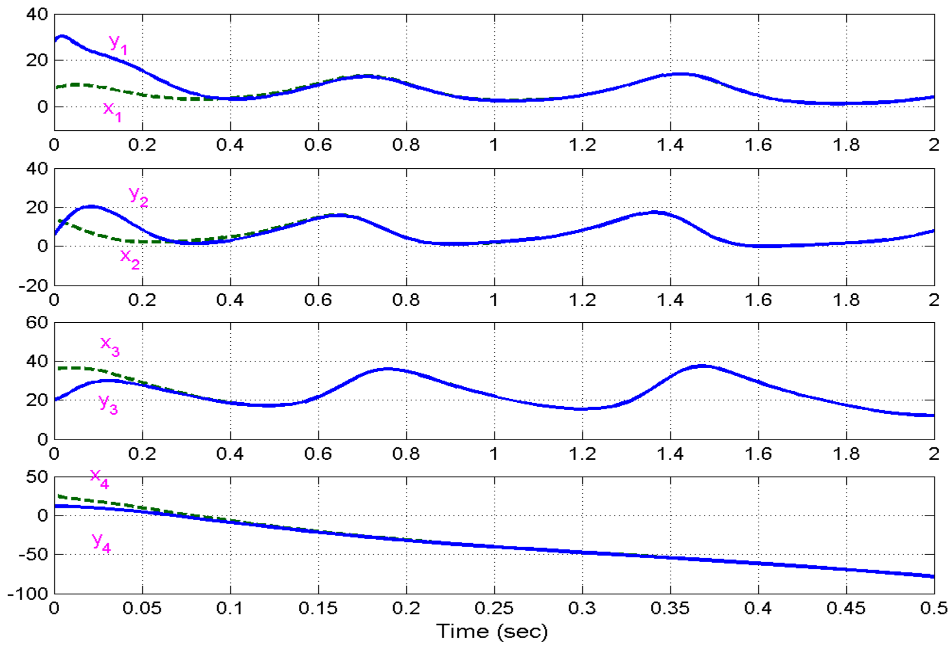

Figure 3. Complete Synchronization of the Hyperchaotic Lorenz Systems

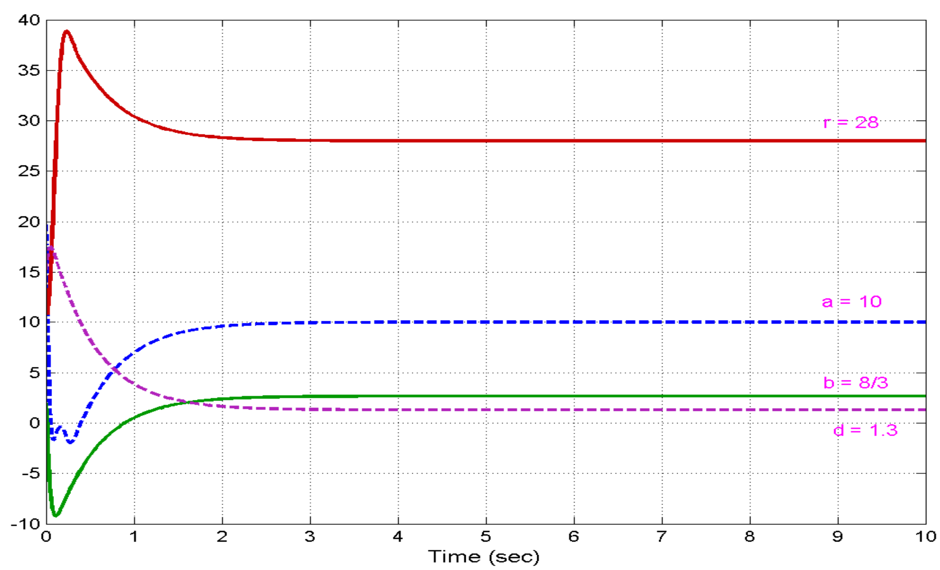

Figure 4. Parameter Estimates $\hat{a}, \hat{b}, \hat{r}, \hat{d}$

4. Adaptive Synchronization of Identical Hyperchaotic Lü Systems

4.1 Theoretical Results

In this section, we discuss the adaptive synchronization of identical hyperchaotic Lü systems ([29], 2006), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Lü dynamics described by

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - x_1) + x_4 \\ \dot{x}_2 &= -x_1 x_3 + \gamma x_2 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \\ \dot{x}_4 &= x_1 x_3 + \delta x_4 \end{aligned} \tag{16}$$

where x_1, x_2, x_3, x_4 are the states and $\alpha, \beta, \gamma, \delta$ are unknown real constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Lü dynamics described by

$$\begin{aligned} \dot{y}_1 &= \alpha(y_2 - y_1) + y_4 + u_1 \\ \dot{y}_2 &= -y_1 y_3 + \gamma y_2 + u_2 \\ \dot{y}_3 &= y_1 y_2 - \beta y_3 + u_3 \\ \dot{y}_4 &= y_1 y_3 + \delta y_4 + u_4 \end{aligned} \tag{17}$$

where y_1, y_2, y_3, y_4 are the states and u_1, u_2, u_3, u_4 are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

$$e_i = y_i - x_i, \quad (i = 1, 2, 3, 4) \tag{18}$$

The error dynamics is easily obtained as

$$\begin{aligned} \dot{e}_1 &= \alpha(e_2 - e_1) + e_4 + u_1 \\ \dot{e}_2 &= \gamma e_2 - y_1 y_3 + x_1 x_3 + u_2 \\ \dot{e}_3 &= -\beta e_3 + y_1 y_2 - x_1 x_2 + u_3 \\ \dot{e}_4 &= \delta e_4 + y_1 y_3 - x_1 x_3 + u_4 \end{aligned} \tag{19}$$

Let us now define the adaptive control functions

$$\begin{aligned} u_1(t) &= -\hat{\alpha}(e_2 - e_1) - e_4 - k_1 e_1 \\ u_2(t) &= -\hat{\gamma} e_2 + y_1 y_3 - x_1 x_3 - k_2 e_2 \\ u_3(t) &= \hat{\beta} e_3 - y_1 y_2 + x_1 x_2 - k_3 e_3 \\ u_4(t) &= -\hat{\delta} e_4 - y_1 y_3 + x_1 x_3 - k_4 e_4 \end{aligned} \tag{20}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\delta}$ are estimates of α, β, γ and δ, respectively, and $k_i, (i = 1, 2, 3, 4)$ are positive constants.

Substituting (20) into (19), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (\alpha - \hat{\alpha})(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= (\gamma - \hat{\gamma}) e_2 - k_2 e_2 \\ \dot{e}_3 &= -(\beta - \hat{\beta}) e_3 - k_3 e_3 \\ \dot{e}_4 &= (\delta - \hat{\delta}) e_4 - k_4 e_4 \end{aligned} \tag{21}$$

Let us now define the parameter estimation errors as

$$e_\alpha = \alpha - \hat{\alpha}, \;\; e_\beta = \beta - \hat{\beta}, \;\; e_\gamma = \gamma - \hat{\gamma} \;\text{ and }\; e_\delta = \delta - \hat{\delta}. \tag{22}$$

Substituting (22) into (21), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_\alpha (e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= e_\gamma e_2 - k_2 e_2 \\ \dot{e}_3 &= -e_\beta e_3 - k_3 e_3 \\ \dot{e}_4 &= e_\delta e_4 - k_4 e_4 \end{aligned} \tag{23}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V(e_1, e_2, e_3, e_4, e_\alpha, e_\beta, e_\gamma, e_\delta) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2\right) \tag{24}$$

which is a positive definite function on R^8.

We also note that

$$\dot{e}_\alpha = -\dot{\hat{\alpha}}, \;\; \dot{e}_\beta = -\dot{\hat{\beta}}, \;\; \dot{e}_\gamma = -\dot{\hat{\gamma}} \;\text{ and }\; \dot{e}_\delta = -\dot{\hat{\delta}} \tag{25}$$

Differentiating (24) along the trajectories of (23) and using (25), we obtain

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_\alpha \left[e_1 (e_2 - e_1) - \dot{\hat{\alpha}} \right] \\ &+ e_\beta \left[-e_3^2 - \dot{\hat{\beta}} \right] + e_\gamma \left[e_2^2 - \dot{\hat{\gamma}} \right] + e_\delta \left[e_4^2 - \dot{\hat{\delta}} \right] \end{aligned} \tag{26}$$

In view of Eq. (26), the estimated parameters are updated by the following law:

$$\begin{aligned}\dot{\hat{\alpha}} &= e_1(e_2 - e_1) + k_5 e_\alpha \\ \dot{\hat{\beta}} &= -e_3^2 + k_6 e_\beta \\ \dot{\hat{\gamma}} &= e_2^2 + k_7 e_\gamma \\ \dot{\hat{\delta}} &= e_4^2 + k_8 e_\delta\end{aligned} \tag{27}$$

where k_5, k_6, k_7 and k_8 are positive constants.

Substituting (27) into (26), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_\alpha^2 - k_6 e_\beta^2 - k_7 e_\gamma^2 - k_8 e_\delta^2 \tag{28}$$

which is a negative definite function on R^8.

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error $e_i, (i = 1,2,3,4)$ and the parameter estimation error $e_\alpha, e_\beta, e_\gamma, e_\delta$ decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 2. *The identical hyperchaotic Lü systems (16) and (17) with unknown parameters are globally and exponentially synchronized by the adaptive control law (20), where the update law for the parameter estimates is given by (27) and* $k_i, (i = 1,2,\ldots,8)$ *are positive constants.* ■

4.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (16) and (17) with the adaptive control law (14) and the parameter update law (27) using MATLAB. We take $k_i = 2$ for $i = 1,2,\ldots,8$.

For the hyperchaotic Lü systems (16) and (17), the parameter values are taken as

$$\alpha = 36, \quad \beta = 3, \quad \gamma = 20 \quad \text{and} \quad \delta = 1.3.$$

Suppose that the initial values of the parameter estimates are

$$\hat{\alpha}(0) = 2, \quad \hat{\beta}(0) = 8, \quad \hat{\gamma}(0) = 4, \quad \hat{\delta}(0) = 5$$

The initial values of the master system (16) are taken as

$$x_1(0) = 18, \quad x_2(0) = 24, \quad x_3(0) = 14, \quad x_4(0) = 30.$$

The initial values of the slave system (17) are taken as

$$y_1(0) = 20, \quad y_2(0) = 7, \quad y_3(0) = 32, \quad y_4(0) = 9.$$

Figure 5 depicts the complete synchronization of the identical hyperchaotic Lü systems (16) and (17). Figure 6 shows that the estimated values of the parameters, viz. $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\delta}$ converge to the system parameters $\alpha = 36,\ \beta = 3,\ \gamma = 20$ and $\delta = 1.3.$

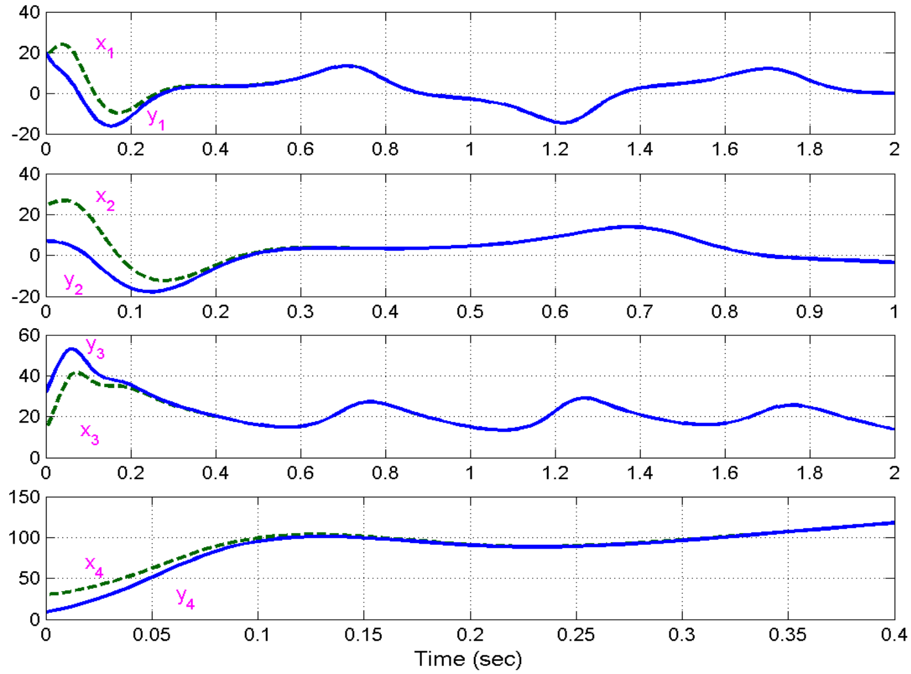

Figure 5. Complete Synchronization of the Hyperchaotic Lü Systems

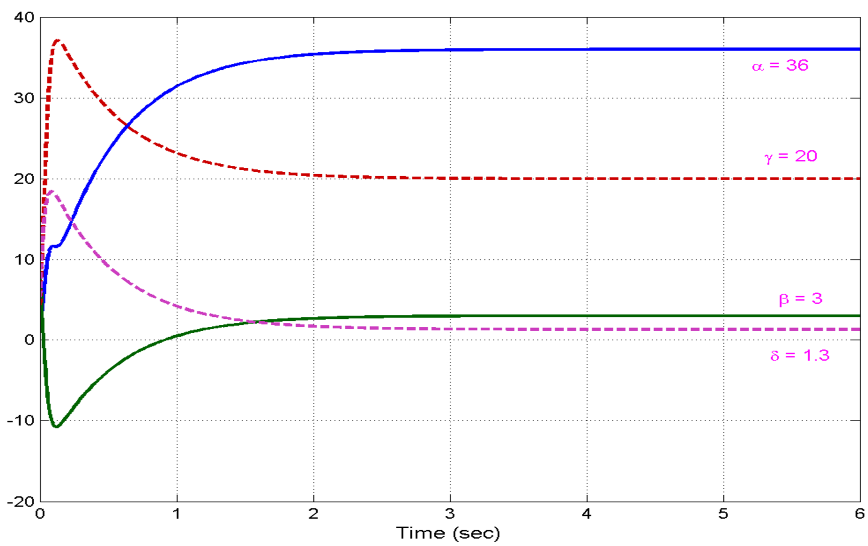

Figure 6. Parameter Estimates $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$

5. ADAPTIVE SYNCHRONIZATION OF NON-IDENTICAL HYPERCHAOTIC LORENZ AND HYPERCHAOTIC LÜ SYSTEMS

5.1 Theoretical Results

In this section, we discuss the adaptive synchronization of non-identical hyperchaotic Lorenz system ([28], 2007) and hyperchaotic Lü system ([29], 2006), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Lorenz dynamics described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_4 \\ \dot{x}_2 &= -x_1 x_3 + r x_1 - x_2 \\ \dot{x}_3 &= x_1 x_2 - b x_3 \\ \dot{x}_4 &= -x_1 x_3 + d x_4 \end{aligned} \tag{29}$$

where x_1, x_2, x_3, x_4 are the states and a, b, c, d are unknown real constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Lü dynamics described by

$$\begin{aligned} \dot{y}_1 &= \alpha(y_2 - y_1) + y_4 + u_1 \\ \dot{y}_2 &= -y_1 y_3 + \gamma y_2 + u_2 \\ \dot{y}_3 &= y_1 y_2 - \beta y_3 + u_3 \\ \dot{y}_4 &= y_1 y_3 + \delta y_4 + u_4 \end{aligned} \tag{30}$$

where y_1, y_2, y_3, y_4 are the states, $\alpha, \beta, \gamma, \delta$ are unknown real constant parameters of the system and u_1, u_2, u_3, u_4 are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

$$e_i = y_i - x_i, \quad (i = 1, 2, 3, 4) \tag{31}$$

The error dynamics is easily obtained as

$$\begin{aligned} \dot{e}_1 &= \alpha(y_2 - y_1) - a(x_2 - x_1) + e_4 + u_1 \\ \dot{e}_2 &= \gamma y_2 + x_2 - r x_1 - y_1 y_3 + x_1 x_3 + u_2 \\ \dot{e}_3 &= -\beta y_3 + b x_3 + y_1 y_2 - x_1 x_2 + u_3 \\ \dot{e}_4 &= \delta y_4 - d x_4 + y_1 y_3 + x_1 x_3 + u_4 \end{aligned} \tag{32}$$

Let us now define the adaptive control functions

$$\begin{aligned} u_1(t) &= -\hat{\alpha}(y_2 - y_1) + \hat{a}(x_2 - x_1) + e_4 - k_1 e_1 \\ u_2(t) &= -\hat{\gamma} y_2 + \hat{r} x_1 - x_2 + y_1 y_3 - x_1 x_3 - k_2 e_2 \\ u_3(t) &= \hat{\beta} y_3 - \hat{b} x_3 - y_1 y_2 + x_1 x_2 - k_3 e_3 \\ u_4(t) &= -\hat{\delta} y_4 - \hat{d} x_4 - y_1 y_3 - x_1 x_3 - k_4 e_4 \end{aligned} \tag{33}$$

where $\hat{a}, \hat{b}, \hat{r}, \hat{d}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\delta}$ are estimates of $a, b, r, d, \alpha, \beta, \gamma$ and δ, respectively, and $k_i, (i = 1, 2, 3, 4)$ are positive constants.

Substituting (33) into (32), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (\alpha - \hat{\alpha})(y_2 - y_1) - (a - \hat{a})(x_2 - x_1) - k_1 e_1 \\ \dot{e}_2 &= (\gamma - \hat{\gamma}) y_2 - (r - \hat{r}) x_1 - k_2 e_2 \\ \dot{e}_3 &= -(\beta - \hat{\beta}) y_3 + (b - \hat{b}) x_3 - k_3 e_3 \\ \dot{e}_4 &= (\delta - \hat{\delta}) y_4 - (d - \hat{d}) x_4 - k_4 e_4 \end{aligned} \tag{34}$$

Let us now define the parameter estimation errors as

$$\begin{aligned} & e_a = a - \hat{a}, \;\; e_b = b - \hat{b}, \;\; e_r = r - \hat{r}, \; e_d = d - \hat{d} \\ & e_\alpha = \alpha - \hat{\alpha}, \;\; e_\beta = \beta - \hat{\beta}, \;\; e_\gamma = \gamma - \hat{\gamma}, \; e_\delta = \delta - \hat{\delta} \end{aligned} \tag{35}$$

Substituting (35) into (32), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_\alpha (y_2 - y_1) - e_a x_2 - x_1) - k_1 e_1 \\ \dot{e}_2 &= e_\gamma y_2 - e_r x_1 - k_2 e_2 \\ \dot{e}_3 &= -e_\beta y_3 + e_b x_3 - k_3 e_3 \\ \dot{e}_4 &= e_\delta y_4 - e_d x_4 - k_4 e_4 \end{aligned} \tag{36}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_r^2 + e_d^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2 \right) \tag{37}$$

which is a positive definite function on R^{12}.

We also note that

$$\dot{e}_a = -\dot{\hat{a}}, \; \dot{e}_b = -\dot{\hat{b}}, \;\; \dot{e}_r = -\dot{\hat{r}}, \dot{e}_d = -\dot{\hat{d}}, \; \dot{e}_\alpha = -\dot{\hat{\alpha}}, \;\; \dot{e}_\beta = -\dot{\hat{\beta}}, \;\; \dot{e}_\gamma = -\dot{\hat{\gamma}}, \;\; \dot{e}_\delta = -\dot{\hat{\delta}} \tag{38}$$

Differentiating (37) along the trajectories of (36) and using (38), we obtain

$$\begin{aligned}\dot{V} = & -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 + e_a\left[-e_1(x_2 - x_1) - \dot{\hat{a}}\right] + e_b\left[e_3x_3 - \dot{\hat{b}}\right] \\ & + e_r\left[-e_2x_1 - \dot{\hat{r}}\right] + e_d\left[-e_4x_4 - \dot{\hat{d}}\right] + e_\alpha\left[e_1(y_2 - y_1) - \dot{\hat{\alpha}}\right] \\ & + e_\beta\left[-e_3y_3 - \dot{\hat{\beta}}\right] + e_\gamma\left[e_2y_2 - \dot{\hat{\gamma}}\right] + e_\delta\left[e_4y_4 - \dot{\hat{\delta}}\right]\end{aligned} \tag{39}$$

In view of Eq. (39), the estimated parameters are updated by the following law:

$$\begin{aligned} \dot{\hat{a}} &= -e_1(x_2 - x_1) + k_5e_a, & \dot{\hat{\alpha}} &= e_1(y_2 - y_1) + k_9e_\alpha \\ \dot{\hat{b}} &= e_3x_3 + k_6e_b, & \dot{\hat{\beta}} &= -e_3y_3 + k_{10}e_\beta \\ \dot{\hat{r}} &= -e_2x_1 + k_7e_r, & \dot{\hat{\gamma}} &= e_2y_2 + k_{11}e_\gamma \\ \dot{\hat{d}} &= -e_4x_4 + k_8e_d, & \dot{\hat{\delta}} &= e_4y_4 + k_{12}e_\delta \end{aligned} \tag{40}$$

where k_5, k_6, k_7 and k_8 are positive constants.

Substituting (40) into (39), we obtain

$$\dot{V} = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 - k_5e_a^2 - k_6e_b^2 - k_7e_r^2 - k_8e_d^2 - k_9e_\alpha^2 - k_{10}e_\beta^2 - k_{11}e_\gamma^2 - k_{12}e_\delta^2 \tag{41}$$

which is a negative definite function on R^{12}.

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error $e_i, (i = 1,2,3,4)$ and the parameter estimation error decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 3. *The non-identical hyperchaotic Lorenz system (29) and hyperchaotic Lü system (30) with unknown parameters are globally and exponentially synchronized by the adaptive control law (33), where the update law for the parameter estimates is given by (40) and* $k_i, (i = 1,2,\ldots,12)$ *are positive constants.* ■

5.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (29) and (30) with the adaptive control law (27) and the parameter update law (40) using MATLAB. We take $k_i = 2$ for $i = 1,2,\ldots,12.$

For the hyperchaotic Lorenz system (29) and hyperchaotic Lü system (30), the parameter values are taken as

$$a = 10,\ \ b = 8/3,\ \ r = 28,\ \ d = 1.3,\ \ \alpha = 36,\ \ \beta = 3,\ \ \gamma = 20 \ \text{ and } \ \delta = 1.3. \tag{42}$$

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0)=7,\ \hat{b}(0)=10,\ \hat{r}(0)=1,\ \hat{d}(0)=4,\ \hat{\alpha}(0)=2,\ \hat{\beta}(0)=8,\ \hat{\gamma}(0)=4,\ \hat{\delta}(0)=5\,.$$

The initial values of the master system (29) are taken as

$$x_1(0)=12,\ \ x_2(0)=10,\ \ x_3(0)=32,\ \ x_4(0)=27.$$

The initial values of the slave system (30) are taken as

$$y_1(0)=4,\ \ y_2(0)=27,\ \ y_3(0)=12,\ \ y_4(0)=16.$$

Figure 7 depicts the complete synchronization of the non-identical hyperchaotic Lorenz and hyperchaotic Lü systems. Figure 8 shows that the estimated values of the parameters, viz. $\hat{a},\hat{b},\hat{r},\hat{d},\hat{\alpha},\hat{\beta},\hat{\gamma}$ and $\hat{\delta}$ converge to the original values of the parameters given in (42).

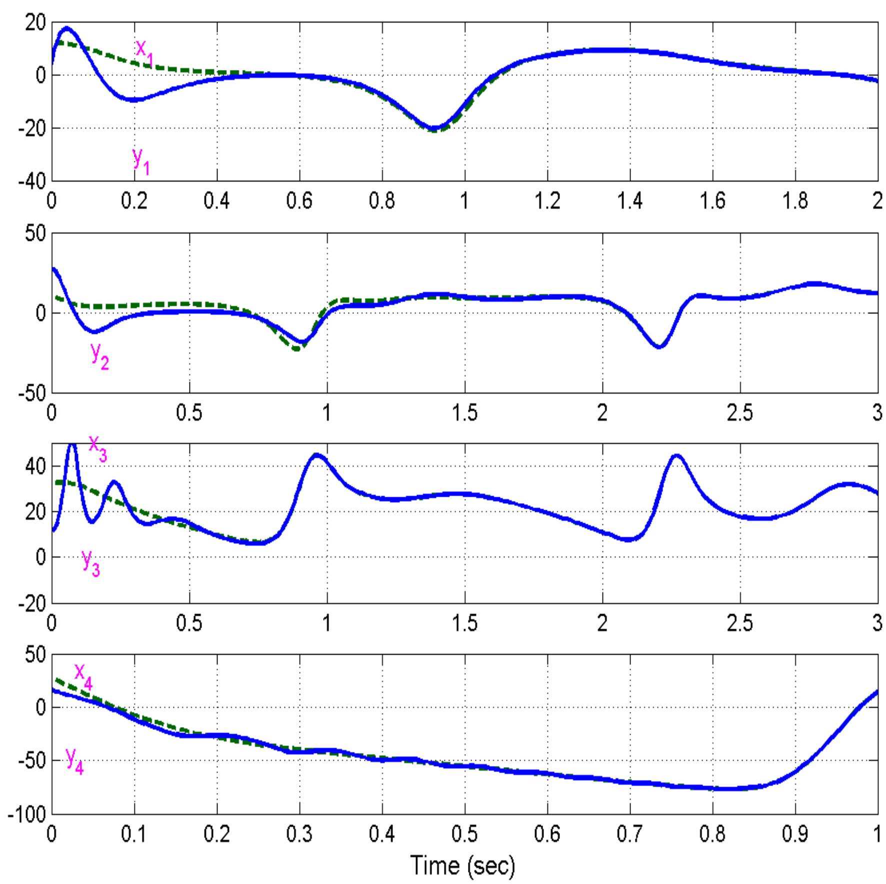

Figure 7. Complete Synchronization of the Hyperchaotic Lorenz and Hyperchaotic Lü Systems

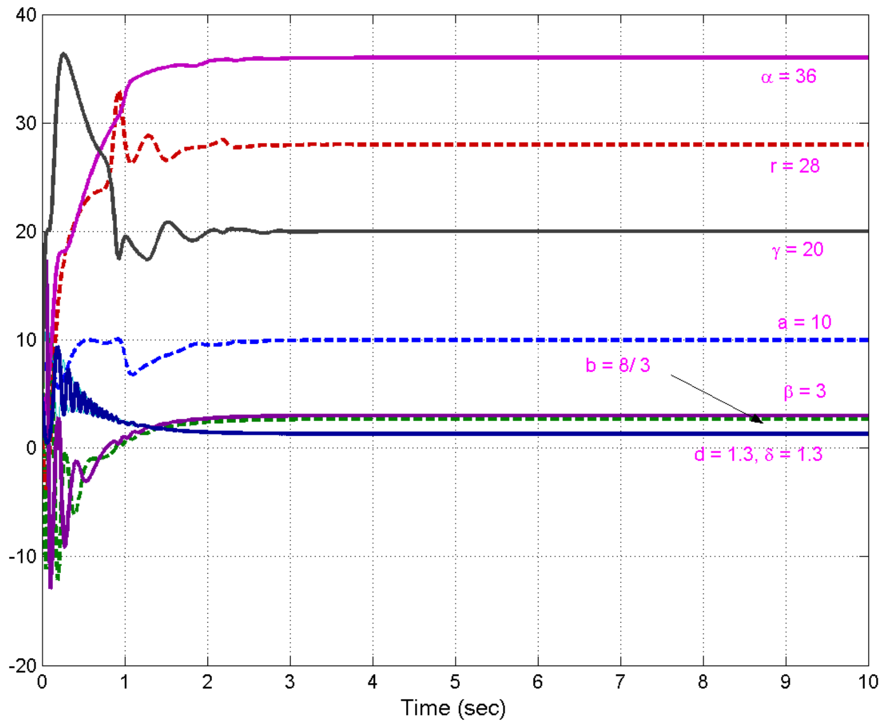

Figure 8. Parameter Estimates $\hat{a}, \hat{b}, \hat{r}, \hat{d}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$

5. CONCLUSIONS

In this paper, we have applied adaptive control method for the global chaos synchronization of identical hyperchaotic Lorenz systems (2007), identical hyperchaotic Lü systems (2006) and non-identical hyperchaotic Lorenz and hyperchaotic Lü systems with unknown parameters. The adaptive synchronization results derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is a very effective and convenient for achieving chaos synchronization for the uncertain hyperchaotic systems discussed in this paper. Numerical simulations are shown to demonstrate the effectiveness of the adaptive synchronization schemes derived in this paper for the synchronization of identical and non-identical uncertain hyperchaotic Lorenz and hyperchaotic Lü systems.

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