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Radio Number Of Wheel Like Graphs

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Abstract.

In this paper we establish the radio number for Flower Wheel graph ($F W^k_n$), k-Wheel graph (kW) and Joint-Wheel graph($W H_n$).

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Radio number, Radio labeling, Flower Wheel graph ($F W^k_n$),k-Wheel graph (kW), Joint-Wheel graph ($W H_n$)

1. INTRODUCTION

A radio labeling is an assignment of labels, traditionally represented byintegers, to the vertices of a graph. Formally, for a given graph G = (V, E)with V being the set of vertices and E being the set of edges, a radio label-ing is a function from the vertices of the graph to some subset of positiveintegers.

For a set of given stations, the task is to assign to each city a channel, which is a non-negative integer, so that interference is prohibited and the span of the channel assigned is minimized. Hale was the first who proposed graph to model these channel assignment in 1980 [5]. Later in 2001 Chartrand, Erwin, Zhang, and Harary were motivated by regulations for channel assignments of FM radio stations to introduce the radio labeling of graphs [1]. Usually, the level of interference between any two stations is closely related to the geographic locations of the station, the closer are the stations the stronger is the interference. Suppose we consider two levels of interference, major and minor. Major interference occurs between two very close stations; to avoid it, the channel assigned to a pair of very close

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stations have to be at least two apart. Manor interference occurs between close stations; to avoid it, the channel assigned to a pair of close stations should be different. To model this problem, we construct a graph G byrepresenting each station by a vertex and connecting two vertices by an edge if the geographical locations of the corresponding stations are very close. Two close stations are represented by, in the corresponding graph G, a pair of vertices that are distance two apart.

For a simple graph G , let diam (G) denote the diameter of G which is themaximum shortest distance between two distinct vertices. For any two vertices u and v in G , let d (u, v) denote the smallest distance between u andv . Radio labeling(multi-level distance labeling or distance labeling) for G isa one-to-one mapping f : V (G) → Z+satisfying the condition

$$d(u, v) + |f(u) - f(v)| \geq 1 + diam(G) \qquad (1.1)$$

for all $u, v \in V(G)$. The span of a labeling f is the maximum integer thatf maps to a vertex of a graph G . The radio number($rn(G)$) of G is thelowest span over all radio labelings of the graph. We will refer to inequality(1.1) as the radio condition. Note that this condition necessitates the useof distance integers, thus $rn(G) \geq |V(G)|$ for all graphs G . Radio labelingare sometimes referred to as multi-distance labeling and they are equivalentto k -labeling for k = diam (G). In this paper we will consider simple andundirected graph.

2. Some Known Results

In this section we recall some known results about the radio number ofgraphs. Chartrand, Erwin, and Zhang [1] gave the upper bound for theradio number of Path(P_n).

Theorem 2.1.[1] For any positive integer n,

$$rn(P_n) \leq \begin{cases} 2k^2 + k, & if \quad n=2k+1; \\ 2(k^2 - k) + 1, & if \quad n=2k, \end{cases}$$

where$P_n$is the Path on n vertices. Moreover, the bound is sharp when$2 \leq n \leq 5$.
The exact value for the radio number of Path was given by Liu, and Zhu[8].

Theorem 2.2. [8] For any n ≥ 4 ,

$$rn(P_n) = \begin{cases} 2k^2 + 2, & if \quad n=2k+1; \\ 2k(k-1) + 1, & if \quad n=2k. \end{cases}$$

Also, Liu and Zhu [8] gave the radio number for Cycle(C_n).

Theorem 2.3. [8] Let$C_n$be an n -vertex Cycle. For n ≥ 3 we have

$$rn(C_n) = \begin{cases} (\frac{n-2}{2})\phi(n) + 1, & if \quad n \cong 0,2 \ (mod4); \\ (\frac{n-1}{2})\phi(n), & if \quad n \cong 1,3 \ (mod4), \end{cases}$$

where

$$\phi(n) = \begin{cases} k+1, & if \quad n=4k+1; \\ k+2, & if \quad n=4k+r \quad for \quad r=0,2,3. \end{cases}$$

However Chartrand, Erwin, Harary, and Zhang [2] obtained different valuesthan Liu and Zhu [8]. They found the lower and upper bound for the radionumber of Cycle(C_n).

Theorem 2.4. [2] For $k \geq 3$,

$$rn(C_n) \leq \begin{cases} k^2, & if \quad n=2k+1; \\ k^2 - k + 1, & if \quad n=2k \end{cases}$$

and

$$rn(C_n) \geq 3\lceil \tfrac{n}{2} - 1 \rceil - 1, \quad for \quad n \geq 6.$$

Liu [7] gave the lower bound for the radio number of Tree*(T_n)*.

Theorem 2.5.[7]If$T_n$isann-vertexrootedtreewithdiameterd.Then

$$rn(T_n) \geq (n-1)(d+1)+1-2w(T_n),$$

wherew(T_n)represent theweight.

Theexactvaluefortheradionumberof Hypercube(Q_n)wasgivenbyR. KhennoufaandO.Togni[6].

Theorem 2.6.[6]Foranypositiveintegern≥1,

$$rn(Q_n) = (2^{n-1} - 1)\lceil \tfrac{n+3}{2} \rceil + 1.$$

M.M.Rivera,M.Tomova,C.Wyels,andA.Yeager[10] gavetheradio numberofCn_Cn,wheredenotetheCartesianproduct.

Theorem 2.7.[10]Foranynon-negativeintegerk,wehave

$$rn(C_{2k} \square C_{2k}) = 2k^3 + 4k^2 - k$$

and

$$rn(C_{2k+1} \square C_{2k+1}) = 2k^3 + 4k^2 + 2k - 1.$$

In[3] C.Fernandez, A.Flores, M.Tomova, and C.Wyelsworkedon find- ingt he radionumber for Completegraph, Stargraph, Complete Bipartite graph, Wheelgraph and Geargraph. They have proved the following results:

- $rn(K_n)=n.$
- $rn(S_n)=n+2.$
- $rn(K_{m,n})=m+n+1.$
- $rn(W_n)=n+2$ *for* $n\geq 5.$
- $rn(G_n)=4n+2$ *for* $n\geq 4.$

M.T.RahimandI.Tomescu[9]investigatedtheradionumberofHelm graph(H_n).Theyprovedthefollowingresult.

Theorem 2.8.[9]Let$H_n$beaHelmgraph.Forn≥5wehave

$$rn(H_n)=4n+2,$$

wherendenotesthenumberofverticesinacycle.

3.New Result

The radio number ofFlower Wheel graph(FW^k_n): Inthissection wewillfind

The radionumber of Flower Wheelgraph(FW^k_n). First of all we will find the lowerbound by examining labels which have minimum distance between them. For an upperbound, we find a specific radiolabeling which gives us span equal to the lowerbound. FlowerWheelgraph consist of kdisjoint copies of Wheelgraph(W_n) meeting in a commonvertex (differentfromhub). The commonvertex of all the copies of Wheel is named as the centralvertex. Its clearthat FW^k_nhas(t+3)k+1verticesanddiam(FW^k_n)=4foralln≥ 5,where n is the number of vertices in onecopy of Wheelgraph. We denote the number of vertices (in one copyofwheel)which a renon-adjacent to the central vertex by t. We consider the case when all the copies of Wheelgraph have same number of vertices.

ThelabelingofFW^k_n isdefinedasfollows:

Toestablishtheradio numberofFW^k_n wewillrefertoalabelingofthevertices{z,v_1,v_2,...,v_{2k}, v_{2k+1},v_{2k+2},...,v_{3k},u_1,u_2,...,u_{tk}}ofFW^k_nthatdistinguishesthe verticesbytheircharacteristics.The centralvertexislabeledasz,theverticesadjacenttozarelabeledsequentiallyby{v_1, v_2, . . . , v_{2k} , v_{2k+1}, v_{2k+2}, . . . , v_{3k} } in clockwise direction. From Figure 2 it is clear that firstly we label {v_1, v_2, . . . , v_{2k} } where v_1 is not the hub ver- tex, and after labeling these verticeswe label {v_{2k+1}, v_{2k+2}, . . . , v_{3k} }(whichare actually the hub vertices). Vertices which are not adjacent to z are labeled sequentially by {u_1, u_2, . . . , u_{tk}} in clockwise direction. We specifyu_1 adjacentto v_1 and v_{2k+1}. The labeling of FW^4 is shown in Figure 2.

First of all we will find the radio number of FW^k_n. Its a special case of FW^k_4 when $t = 0$, where t is number of vertices(in one copy) which are non-adjacent to z. We can follow the above procedure to label the vertices of FW^k_4.

Theorem 3.1. For $k \geq 2$, $rn(FW^k_4) = 3k + 2$.

Proof. First of all we will find the lower bound for the radio number of FW^k_4.

Lowerbound for $rn(FW^k_4)$: Assume $k \geq 2$. Since $diam(FW^k_4) = 2$, so any radio labeling f of FW^k_4 must satisfy the radio condition i.e.

$$d(u, v) + |f(u) - f(v)| \geq 1 + diam(FW^k_n) \geq 3$$

hold for all distinct $u, v \in V(FW^k_4)$. To determine the lower bound we have to count the minimum number of restricted values associated with the vertices of FW^k_4. Let $f(z) = a$, where $a \in Z^+$. Since $d(z, v_i) = 1$, where $z = v_i$ for $1 \leq i \leq 3k$. The radio condition becomes

$d(z, v_i) + |f(z) - f(v_i)| \geq 1 + diam(FW^k_4)$, or $1 + |f(z) - f(v_i)| \geq 3$, or $|f(z) - f(v_i)| \geq 2$. So, there exist one restricted value associated with z. If $d(v_i, v_j) \leq 2$, where $1 \leq i, j \leq 3k$, then the radio condition becomes

$d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 1 + diam(FW^k_4)$, or $2 + |f(v_i) - f(v_j)| \geq 3$, or $|f(v_i) - f(v_j)| \geq 1$. So, we can assign the consecutive integers to the following sets $\{v_1, v_3, \ldots, v_{2k-1}\}$, $\{v_2, v_4, \ldots, v_{2k}\}$ and $\{v_{2k+1}, v_{2k+2}, \ldots, v_{3k}\}$ respectively. Therefor, there exist no restricted value associated with v_i for $1 \leq i \leq 3k$. Hence, there is only one restricted value associated with any label of FW^k_4. Thus, $rn(FW^k_4) \geq$ allowed values+restricted value

Hence, $rn(FW^k_4) \geq 3k + 1 + 1 = 3k + 2$.

Upper bound for $rn(FW^k_4)$: If f is any radio labeling of FW^k_4, then span of this labeling will provide an upper bound for the radio number of FW^k_4. In order to find an upper bound we define a radio labeling $f : V(FW^k_4) \rightarrow Z^+$ as follows:

$f(z) = 1,$

$f(v_{2i-1}) = 2 + i, \quad$ for $\quad 1 \le i \le k,$

$f(v_{2i}) = 2 + k + i, \quad$ for $\quad 1 \le i \le k,$

$f(v_{2k+i}) = 2(1 + k) +$ for $\quad 1 \le i \le k.$

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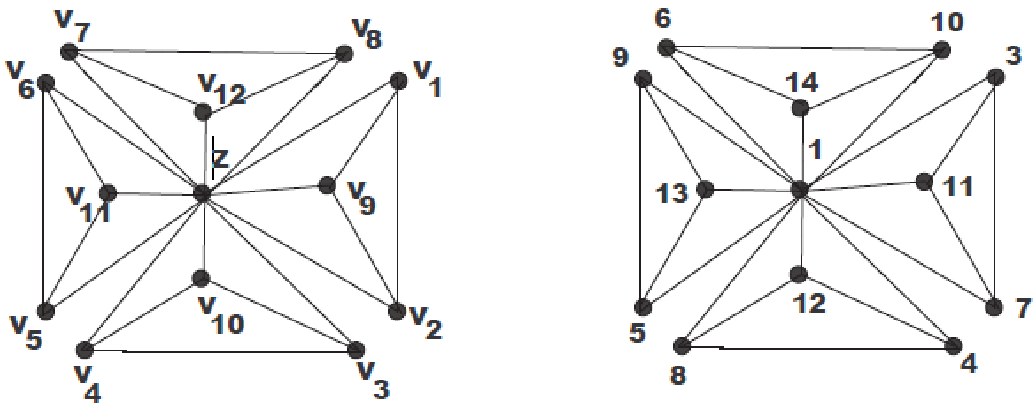


Figure 1. Radio labeling of $FW^4{}_4$

Claim: The labeling f is a valid radio labeling. We have to show that the radio condition

$d(u, v) + |f(u) - f(v)| \ge 1 + diam(FW_4^k) \ge 3$

holds for all distinct $u, v \in V(FW_4^k)$. We will discuss two cases:

Case1: Since $d(z, v_i) = 1$, where $1 \le i \le 3k$ and $f(z) = 1, f(v_i) \ge 3$. The radio condition in this case will be

$d(z, v_i) + |f(z) - f(v_i)| \ge 1 + |1 - 3|, or\ d(z, v_i) + |f(z) - f(v_i)| \ge 3.$

Hence, the radio condition is satisfied.

Case2: Since $d(v_i, v_j) \le 2$, where $1 \le i, j \le 3k$ and $f(v_i) \ge 3$. The possible label difference for each pair will satisfy $|f(v_i) - f(v_j)| \ge 1$. The radio condition in this case will be

$$d(v_i, v_j) + |f(v_i) - f(v_j)| \ge 2 + 1, or\ d(v_i, v_j) + |f(v_i) - f(v_j)| \ge 3.$$

Hence, the radio condition is satisfied.

These two cases establish the claim that f is a valid radio labeling of FW_4^k.

Thus, $rn(FW_4^k) \le span(f) = 3k + 2.$

From the lower and upper bound of $rn(FW_4^k)$, we have

$rn(FW^k{}_4) = 3k + 2.$

An example of radio labeling of FW^4_4 is shown in Figure 1.
In the next theorem we will find the lower bound for the radio number of FW^k_n.

Theorem 3.2. *For k ≥ 4 and n ≥ 5,*

$$rn(FW^k_n) \geq tk + 9k + 2,$$

wherek is the number of copies of Wheel, n is the number of vertices in each copy of the Wheel and t be the number of vertices which are non-adjacent to the central vertex.

Proof. Assume k ≥ 4. Since diam(FW^k_n) = 4, so any radio labeling f of FW^k_n must satisfy the radio condition i.e.

$$d(u, v) + |f(u) - f(v)| \geq 5$$

holds for all distinct u, v ∈ V(FW^k_n). Now we count the total number ofrestricted values:

Restricted values associated with any label of z: If z is label as a i.e. f(z) = a, then as d(z, u_i) = 2 for 1 ≤ i ≤ tk, where z = u_i for all u_i non-adjacent with z, the radio condition becomes
d(z, u_i)+|f(z)−f(u_i)| ≥ 1+diam(FW^k_n), or 2+|f(z)−f(u_i)| ≥ 1+ 4, or |f(z) − f(u_i)| ≥ 3. Hence, the number of restricted values associatedwith any label of z are 2.

Restricted value associated with any label of the vertices non- adjacent to z: Since d(u_i, u_j) ≤ 4, when i = j and for all 1 ≤ i, j ≤ tk−1.The radio condition becomes

d(u_i, u_j) + |f(u_i) − f(u_j)| ≥ 5,or 4 + |f(u_i) − f(u_j)| ≥ 5,or |f(u_i) − f(u_j)| ≥ 1. It means we can assign consecutive integers to u_i,which implies that there are no restricted value associated with any labelof u_i.

Restricted value associated with any label of the vertex u_{tk} non- adjacent to z: Suppose d(u_{tk}, v_i) ≤ 3 for 1 ≤ i ≤ 3k, where u_{tk}= v_i andfor all v_i adjacent to z, the radio condition in this case will be

d(u_{tk}, v_i) + |f(u_{tk}) − f(v_i)| ≥ 5,or 3 + |f(u_{tk}) − f(v_i)| ≥ 5,or |f(u_{tk}) − f(v_i)| ≥ 2. So, there is only one restricted value corresponding to u_{tk}.

Restricted values associated with any label of the vertices ad- jacent to the central vertex: Since v_i denote any vertex adjacent to z.If d(v_i, v_j) ≤ 2, when v_i= v_j for 1 ≤ i, j ≤ 3k. Then, the radio conditionbecomes
d(v_i, v_j) + |f(v_i) − f(v_j)| ≥ 1 + 4,or 2 + |f(v_i) − f(v_j)| ≥ 5,or |f(v_i) − f(v_j)| ≥ 3. Therefore, restricted values associated with each label of v_i are2. Since we have two restricted values for each 3k−1 vertices. Hence, the total restricted values in this case will be 2(3k − 1).

Total number of restricted values associated with any label of FW^k_n:

Total number of restricted values associated with any label of $FW^k{}_n$ will be the sum of restricted value associated with z+ restricted value associated with u_i+ restricted value associated with u_{tk}+ restricted value associatedwith $v_i = 2 + 0 + 1 + 2(3k - 1) = 6k + 1$

Hence, rn(F W^k_n) ≥ allowed values + restricted values

$$=(t + 3)k + 1 + 6k + 1,$$
$$=tk + 9k + 2.$$

Hence, we establish the lower bound for the radio number of $FW^k{}_n$.

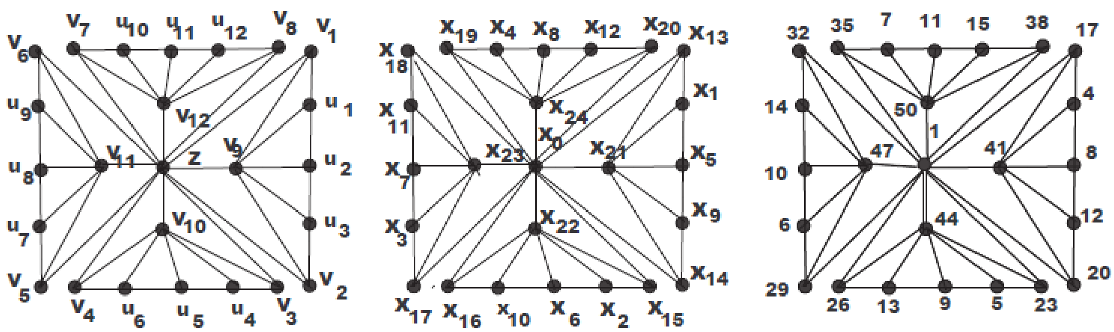


Figure 2.Relabeling and Radio labeling of FW^4_7

Our next result will give the upper bound for the radio number of $FW^k{}_n$.

Theorem 3.3. For $k \geq 4$ *and* $n \geq 5$, $rn(FW^k_n) \leq tk + 9k + 2$.

Proof. If f is any radio labeling of FW^k_n; then span of this labeling will provide an upper bound for the radio number of $FW^k{}_n$. In order to find an upper bound firstly we define the position function p that renames the vertices of FW_n^k using the set $\{x_0, x_1, \ldots, x_{(t+3)k}\}$. Then we specify the labels $f(x_i)$ so that $i < j$ if and only if $f(i) < f(j)$. For $k \geq 2$ and $n \geq 5$, the position function $p : V(FW_n^k) \rightarrow \{x_0, x_1, \ldots, x_{(t+3)k}\}$ is defined as follows:

$p(z)$=x_0.

For $1 \leq j \leq t$, $\quad p(u_{j+t(i-1)}) = x_{i+(j-1)k}$, where $1 \leq i \leq k$

and $p(v_i)$=x_{tk+i} for $1 \leq i \leq 3k$.

Next, we define a radio labeling $f : \{x_0, x_1, \ldots, x_{(t+3)k}\} \rightarrow Z^+$ as follows:

$$f(x_i) = \begin{cases} 1, & for \quad i = 0; \\ 3 + i, & for \quad 1 \leq i \leq tk; \\ tk + 2 + 3(i - tk), & for \quad tk + 1 \leq i \leq (t+3)k. \end{cases}$$

Claim: The labeling $\mathbf{f}$ is a valid radio labeling. Wehave to show that the radio condition

$$d(u, v) + |\mathbf{f}(u) - \mathbf{f}(v)| \geq 1 + diam(FW^k_n) \geq 5$$

must holds for all pair of vertices (u, v), where u = v.

Case 1: Consider the pair (z, r), when z = **r** for all **r** ∈V (FW^k_n). Since

d(z, **r**) ≤ 2, p(z) = x_0 and p(**r**)= $x_{\mathbf{i}}$for 1 ≤ **i** ≤ (**t** + 3)k. Therefore,$\mathbf{f}(x_{\mathbf{i}})$≥ 4 for all 1 ≤ **i** ≤ (**t** + 3)k and **f** (z) = 1. So, the radio conditionbecomes

$d(z, r) + |f(z) - f(r)| \geq 2 + |1 - 4|$, or $d(z, r) + |f(z) - f(r)| \geq 5$.

Hence, the radio condition is satisfied.

Case 2: Consider the pair of vertices ($v_{\mathbf{i}}$,$v_{\mathbf{j}}$), where 1 ≤ i, **j** ≤ 3k. As$d(v_{\mathbf{i}}, v_{\mathbf{j}}) \leq 2$, the label difference for each pair will be

$|f(v_i)-f(v_j)| = |f(x_{tk+i})-f(x_{tk+j})| = |tk+2+3(i-tk) -tk-2-3(j-tk)|$

$|f(v_i)-f(v_j)| = 3|i-j| \geq 3$. The radio condition becomes

$d(v_{\mathbf{i}}, v_{\mathbf{j}}) + |\mathbf{f}(v_{\mathbf{i}}) - \mathbf{f}(v_{\mathbf{j}})| \geq 2 + 3 = 5$.

Hence, the radio condition is satisfied.

Case 3: Since $d(u_{\mathbf{i}}, u_w) \leq 4$ for $1 \leq i, w \leq tk$, therefore

$|f(u_{j+t(i-1)}) - f(u_{j+t(w-1)})| = |f(x_{i+(j-1)k}) - f(x_{w+(j-1)k})|$

$= |3 + i + (j-1)k - 3 - w - (j-1)k|$

$|f(u_i)-f(u_w)| = |i-w| \geq 1$.Hence, the radio condition becomes

$d(u_i, u_w) + |f(u_i) - f(u_w)| \geq 4 + 1 = 5$.

Hence, the radio condition is satisfied.

Case 4: Consider the pair (v_i, u_w), where $i = w$. As $d(v_i, u_w) \leq 3$ for $1 \leq i \leq 3k$ and $1 \leq w \leq tk$. We have $f(u_w) \in \{4, 5, \ldots, tk + 3\}$ and $f(v_i) \in \{tk + 5, tk + 8, \ldots, tk + 9k + 2\}$. The possible label difference for each pairs are,

$|f(v_i) - f(u_w)| = |tk + 5 - tk - 3| = 2,$
$|f(v_i) - f(u_w)| = |tk + 9k + 2 - 4| = tk + 9k - 2.$
So, $|f(v_i) - f(u_w)| \geq 2$. The radio condition becomes
$d(v_i, u_w) + |f(v_i) - f(u_w)| \geq 3 + 2 = 5.$
Hence, the radio condition is satisfied. These four cases establish the claim that f is a valid radio labeling of FW^k_n.
Thus, $rn(FW^k_n) \leq span(f) = tk + 9k + 2.$

An example of radio labeling of FW^4_7 is shown in Figure 2.
Combing Theorem 3.2 and Theorem 3.3 we have.

Theorem 3.4. The radio number of FW^k_n is $tk + 9k + 2$, when $k \geq 4$ and $n \geq 5$.
Note: It is easy to see that we get the same radio number of FW^k_n for $k = 2$ and $k = 3$ as given in Theorem 3.3 but we cannot follow the above procedure.

The radio number of k-Wheel graph(kW): In this section we will find the radio number of k-Wheel graph(kW) defined as follows: For $k = 1$ we have 1-Wheel graph which is isomorphic to Wheel graph and its radio number is given by [3], for $k \geq 2$ consider k concentric cycles of arbitrary length

and join each vertex of the concentric cycles with the center(K_1). The re- sulting graph denoted by kW is isomorphic to $\{C_1 C_2 \cdots C_k\} + K_1$, where K_1 is a complete graph having one vertex. Its easy to see that number of vertices in kW are $l_i + 1$, where $l_1 \geq l_2 \geq \cdots \geq l_k$ denote the length of the cycles C_i for $1 \leq i \leq k$ respectively and diam(kW) $= 2$. The labeling of kW is defined as follows: The central vertex(hub) is labeled as z, the vertices adjacent to the center are labeled sequentially by $\{v_1, v_2, \ldots, v_k\}$. We start labeling from the outer most cycle which has largest length(l_1). An example for 3-Wheel graph(3W) is shown in Figure 3. We denote number of concentric cycles by k. In the next theorem we will determine the lower bound for rn(kW).

Theorem 3.5. For $k \geq 2$, we have

$$rn(kW) \geq \sum_{i=1}^{k} l_i + 2,$$

Where l_i are the length of concentric cycles.

Proof. Since diam(kW) = 2 for any positive integer k. We have the radio condition $d(u, v) + |f(u) - f(v)| \geq 3$ for all distinct $u, v \in V(kW)$, where f is the radio labeling of kW . First of all we will count the minimum number of restricted labels which will eventually give us the lower bound for the radio number of kW .

Restricted value associated with z: Let us take $f(z) = b$, where $b \in Z^{+}$. Since $d(z, v_j) = 1$ for all $z = v_j$, where $1 \leq j \leq l_i$, then $b+1$ is the restricted label associated with z.

Restricted value associated with v_j: As $d(v_j, v_w) \leq 2$ for all $v_j = v_w$, where $1 \leq j, w \leq l_i$, we will have two cases.

When $l_1 = l_2 = \cdots = l_k$: There exist no restricted value associated with v_j, where $1 \leq j \leq l_i$ i.e. we can assign the consecutive integers to v_j.

When $l_1 \geq l_2 \geq \cdots \geq l_k$: There exist no restricted value associated with v_j, where $1 \leq j \leq l_i$ i.e. we can assign the consecutive integers to v_j.

So, there exist only one restricted value associated with any label of kW .
The total number of allowed labels are,

$l_i + 1$ *for* $l_1 = l_2 = \cdots = l_k$ *and* $l_1 \geq l_2 \geq \cdots \geq l_k$.
Hence, the radio number of kW ≥ allowed values + restricted value

i.e. rn(kW) ≥

which establish the lower bound for rn(kW).
Our next theorem will give the upper bound for rn(kW).

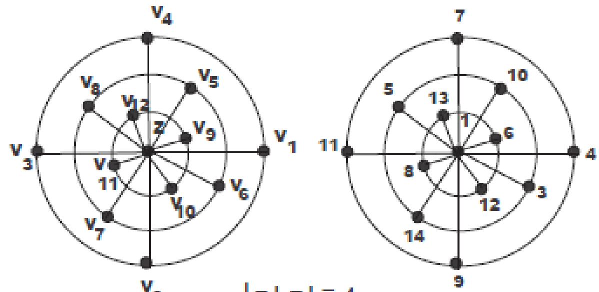
Theorem 3.6. For $k \geq 2$, we have

where l_i are the length of the concentric cycles.

Proof. We will define our radio labeling $f : V(kW) \to Z^{+}$ which will have a minimum span = $^{k}l_i + 2$ is defined as follows:

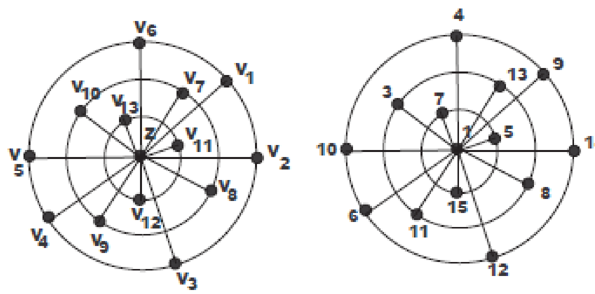
Step1: We start labeling from the central vertex z. Let $f(z) = 1$.

Step2: After labeling z we move to v_j, where $1 \leq j \leq l_i$. We can start labeling from any v_j let $f(v_j) = 3$. In order to label v_j we will consider those vertices which has distance two between them i.e. if $d(v_j, v_w) = 2$, where $1 \leq j, w \leq l_i$ and $j = w$ then we can assign consecutive integer to v_j and v_w so that the radio condition is satisfied. For k = 3 the radio labeling of kW defined above is illustrated in Figure 3.



$l_1 = l_2 = l_3 = 4$

k^{th} \ l_i	1	2	3	4
1st	v_1 4	v_2 9	v_3 11	v_4 7
2nd	v_5 10	v_6 3	v_7 14	v_8 5
3rd	v_9 6	v_{10} 12	v_{11} 8	v_{12} 13



$l_1 = 6, l_2 = 4, l_3 = 3$

k^{th} \ l_i	1	2	3	4	5	6
1st	v_1 9	v_2 14	v_3 12	v_4 6	v_5 10	v_6 4
2nd	v_7 13	v_8 8	v_9 11	v_{10} 3		
3rd	v_{11} 5	v_{12} 15	v_{13} 7			

Figure 3. Radio labeling of 3W

Claim: f is a radio labeling. We must show that the radio condition $d(u, v) + |f(u) - f(v)| \geq diam(G) + 1 \geq 3$ holds for all pair of vertices (u, v) (where u = v). We will have two cases:

Case 1: Consider the pair (z, v_j), where $z = v_j$ for $1 \leq j \leq l_i$. Since $d(z, v_j) = 1$, $f(z) = 1$ and $f(v_j) \geq 3$ for all $1 \leq j \leq l_i$. Examining the label difference for each pair, we have $|f(z) - f(v_j)| \geq 2$. So, the radio condition becomes $d(z, v_j) + |f(z) - f(v_j)| \geq 1 + 2 = 3$. Hence, the radio condition is satisfied in this case.

Case 2: Consider the pair (v_j, v_w), where $(j = w)$ and $1 \leq j, w \leq_{i=1} l_i$. Since $f(v_j) \geq 3$ and $d(v_j, v_w) \leq 2$. So, the label difference will be $|f(v_j) - f(v_w)| \geq 1$ for all distinct v_j, v_w. The radio condition for such pair of vertices becomes $d(v_j, v_w) + |f(v_j) - f(v_w)| \geq 2 + 1 = 3$. Hence, the radio condition is satisfied.

These two cases establish the claim that f is a valid radio labeling of kW. Thus, $rn(kW) \leq span(f) = l_i + 2$.

Note: In Figure 3 when $l_1 = l_2 = l_3 = 4$ we start labeling from v_6 i.e. $f(v_6) = 3$. After labeling v_6 we move to v_1 because $d(v_6, v_1) = 2$ so, we can assign the consecutive integer to v_1 i.e. $f(v_1) = 4$. After labeling v_1 we move to v_8 because $d(v_8, v_1) = 2$ so, we can assign the consecutive integer to v_8 i.e. $f(v_8) = 5$. We continue in the same way and label all the vertices of kW. Similarly, when $l_1 = 6$, $l_2 = 4$ and $l_3 = 3$ we start labeling from v_{10} i.e. $f(v_{10}) = 3$. After labeling v_{10} we move to v_6 because $d(v_{10}, v_6) = 2$ so, we can assign the consecutive integer to v_6 i.e. $f(v_6) = 4$. After labeling v_6 we move to v_{11} because $d(v_{11}, v_6) = 2$ so, we can assign the consecutive integer to v_{11} i.e. $f(v_{11}) = 5$. We continue in the same way and label all the vertices of kW.

Theorem 3.7. If kW is a k-Wheel graph, then

$$rn(kW) = \sum_{i=1}^{k} l_i + 2,$$

where l_i are the length of the concentric cycles.
Proof. Follows from Theorem 3.5 and Theorem 3.6.

The radio number of Joint-Wheel graph(WH_n): Joint-Wheel graph (WH_n) is defined as follows: It consist two disjoint copies of Wheel which are joined by an edge between two rim vertices. It is easy to note that WH_n has $2n + 2$ vertices and $4n + 1$ edges, where n is the number of rim vertices in one copy of the Wheel graph. Its easy to see that for $n \geq 4$, diam(WH_n) = 5.

The labeling of Joint-Wheel is defined as follows:

To establish the radio number of Joint-Wheel we will define a labeling for the vertices of WH_n that distinguishes the vertices by their characteristics. The hub vertices are labeled as z_1 and z_2, the vertices adjacent to z_1 and z_2 are labeled sequentially by $\{v_1, v_2, \ldots, v_n\}$ in counterclockwise direction and by $\{u_1, u_2, \ldots, u_n\}$ in clockwise direction respectively. We specify that v_1, v_{n-1} are adjacent to v_n and u_1, u_{n-1} are adjacent to u_n, also v_n and u_n are the end vertices of the bridge(between two copies of Wheel graph).

Theorem 3.8. For every $n \geq 10$,

$$rn(WH_n) \geq 4n + 7.$$

Proof. Since diam(WH_n) = 5, we must show that the radio condition $d(u, v) + |f(u) - f(v)| \geq 6$ holds for every two distinct vertices u, v ∈ V(WH_n). We start labeling from the vertices v_2 and u_2. If we assume that $f(v_2) = a$ and $f(u_2) = a+1$. Then it may be noted that whenever we assign an integer to one copy of Wheel we must assign the next possible integer to the second copy of Wheel. We will discuss even and odd cases separately.

When n is even: Since we start labeling from v_2 and u_2 i.e. $f(v_2) = a$ and $f(u_2) = a + 1$. So, there exist no restricted value associated with v_2. After labeling u_2 we move to v_i, where $1 \leq i \leq n$ and $i = 2$, if $f(v_n) = a + 4$ then there exist two restricted values associated with u_2. After assigning label to v_n we label u_3 i.e. $f(u_3) = a + 7$. So, there are two restricted val- ues associated with v_n. Following in the similar way we can see that there exist two restricted values associated with each vertex of the following set $\{u_2, v_n, v_3, u_n, u_4\}$.

Consider the pair (v_{2i+2}, u_{2j-1}). Since $d(v_{2i+2}, u_{2j-1}) \leq 5$, where $2 \leq i \leq 1 \leq j \leq 2$ and $j = 2$. If $f(v_6) = b$(the value of b must be great than previously assign integer) then $f(u_1) = b + 2$ i.e. $b + 1$ is the restricted value for the remaining v_i and u_j, where $3 \leq i \leq$ and $3 \leq j \leq \frac{n}{}$.

So, there exist one restricted value associated with v_6. Now we will moveto the first copy of wheel. After labeling$f(u_1) = b + 2$ we assign $b + 4$to v_8 i.e. there exist one restricted value associated ithu_1. Following inthe similar way we can see that there exist one restricted value associatedwith each$v_{2i+2}$and u_{2j-1}, where 2 and $j = 2$.Therefor, total number of restricted values associated with v_{2i+2}areand restricted values associated with $u_{2j-1}$are. Similarly we consider the pair of vertices restricted values associated with $u_{2j+2}$are

Since $d(u_{n-2}, z_1) = 4$, the radio condition becomes

$d(u_{n-2}, z_1) + |f(u_{n-2}) - f(z_1)| \geq 6$, or $|f(u_{n-2}) - f(z_1)| \geq 2$,

whichimplies that there exist two restricted values associated with u_{n-2}. If z_1 is labeled as c(the value of c must be great than previously assigninteger), then any positive value from the set $\{c+1, c+2\}$ assigned to $z_2$will not satisfy the radio condition which is defined above for the pair of vertices(z_1, z_2). So, there are two restricted values associated with z_1.

Therefor, the total number of restricted values will be the sum of restricted values associated with $\{u_2, v_n, v_3, u_n, u_4\}$+ restricted values associated with v_{2i+2}+restricted values associated with u_{2j-1}+ restricted values associated with v_{2i-1}+restricted values associated with u_{2j+2}+restricted values asso- ciated with u_{n-2}+restricted values associated with z_1

When nis odd: Since we start labeling from v_2 and u_2 i.e. $f(v_2) = a$ and$f(u_2) = a + 1$. So, there exist no restricted value associated with v_2.After labeling u_2 we move to v_i,where $1 \leq i \leq n$ and $i = 2$, if $f(v_n) = a + 4$ thenthere exist two restricted values associated with u_2. After assigning label to v_n welabel u_3 i.e. $f(u_3) = a + 7$. So, there are tworestricted values associated with v_n. Following in the similar way we can see that there exist two restricted values associated with each vertex of the following set$\{u_2, v_n, v_3, u_n, u_4\}$.

Consider the pair.(v_{2i+2}, u_{2j-1}). Since $d(v_{2i+2}, u_{2j-1}) \leq 5$ $\frac{n-1}{2}$ $\frac{n-1}{2}$ where $2 \leq i$ $_2$≤than previously $\frac{n-1}{2}$ assign integer) then $f(u_1) = b + 2$ i.e. $b + 1$ is the restrictedvalue for the remaining $v_i$and u_j, where $3 \leq i \leq$ and $3 \leq j \leq$ so, there exist one restricted value associated with v_6. Now we will moveto the first copy of wheel. After labeling $f(u_1) = b + 2$ we assign $b + 4$ tov_8 i.e. there exist one restricted value associated with u_1. Following in thesimilar way we can see that there exist one restricted value associated witheach v_{2i+2}andu_{2j-1}, where $2 \leq i \leq$and $j = 2$. Therefor, total number of restricted values associated with $v_{2i+2}$are andrestricted values associated with $u_{2j-1}$are$_2$. Similarly we consider thepair of vertices (v_{2i-1}, u_{2j+2}), where $1 \leq i \leq_2$ and $i = 2$, $2 \leq j \leq$applying the above procedure we can found the restricted values associatedwith $v_{2i-1}$and u_{2j+2}. Therefor, total number of restricted values associatedwith $v_{2i-1}$are and restricted values associated with $u_{2j+2}$are

Since $d(u_{n-1}, z_1) = 3$, the radio condition becomes

$$d(u_{n-1}, z_1) + |f(u_{n-1}) - f(z_1)| \geq 6, \ \ or \ \ |f(u_{n-1}) - f(z_1)| \geq 3,$$

whichimplies that there exist two restricted values associated with u_{n-1}. If z_1 is labeled as c(the value of c must be great than previously assigninteger), then any positive value from the set $\{c+1, c+2\}$ assigned to $z_2$will not satisfy the radio condition which is defined above for the pair of

vertices(z_1, z_2). So, there are two restricted values associated with z_1.

Therefor, the total number of restricted values will be the sum of restricted values associated with $\{u_2, v_n, v_3, u_n, u_4\}+$ restricted values associated withv_{2i+2}+restricted values associated with $u_{2j-1}+$ restricted values associated with v_{2i-1}+restricted values associated with u_{2j+2}+restricted values asso- ciated with u_{n-1}+restricted values associated with z_1

Hence, $rn(WH_n) \geq$ allowedvalues+restricted values

$$= 2n + 2 + 2n + 5, rn(WH_n) \geq 4n + 7.$$

Theorem 3.9. For $n \geq 10$,

$$rn(WH_n) \leq 4n + 7.$$

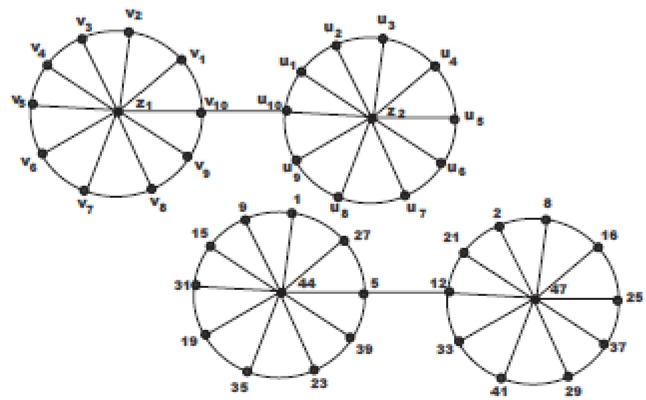
Proof. We provide a radio labeling f of WH_n for $n \geq 10$. The span of thislabeling will provide an upper bound for the radio number of WH_n.Starting with any copy of the Wheelsubgraph of WH_n. Radio labeling$f : V(WH_n) \to Z^+$ is defined as follows:

When nis even:

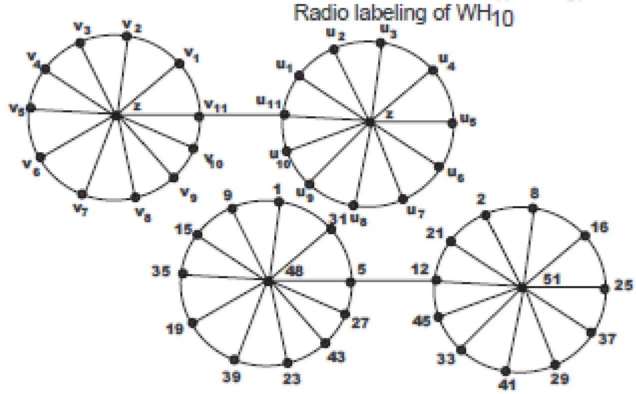
$$\begin{aligned}
&p(v_2) = 1, \quad p(v_3) = 9, \quad p(v_n) = 5, \quad p(v_1) = 2n+7,\\
&p(u_2) = 2, \quad p(u_3) = 8, \quad p(u_4) = 16, \quad p(u_1) = 21, \quad p(u_n) = 12,\\
&p(v_{2i+2}) = 15 + 4(i-1), \qquad for \quad 1 \leq i \leq \tfrac{n-4}{2},\\
&p(v_{2i+3}) = (2n+11) + 4(i-1), \qquad for \quad 1 \leq i \leq \tfrac{n-4}{2},\\
&p(u_{2i+3}) = 25 + 4(i-1), \qquad for \quad 1 \leq i \leq \tfrac{n-4}{2},\\
&p(u_{2i+4}) = (2n+17) + 4(i-1), \qquad for \quad 1 \leq i \leq \tfrac{n-6}{2},\\
&p(z_1) = 4n+4,\\
&p(z_2) = 4n+7.
\end{aligned}$$

When nis even:

$p(v_2) = 1, \quad p(v_3) = 9, \quad p(v_n) = 5, \quad p(v_1) = 2n + 9,$
$p(u_2) = 2, \quad p(u_3) = 8, \quad p(u_4) = 16, \quad p(u_1) = 21, \quad p(u_n) = 12,$
$p(v_{2i+2}) = 15 + 4(i - 1), \qquad for \quad 1 \leq i \leq \frac{n-3}{2},$
$p(v_{2i+3}) = (2n + 13) + 4(i - 1), \qquad for \quad 1 \leq i \leq \frac{n-5}{2},$
$p(u_{2i+3}) = 25 + 4(i - 1), \qquad for \quad 1 \leq i \leq \frac{n-5}{2},$
$p(u_{2i+4}) = (2n + 15) + 4(i - 1), \qquad for \quad 1 \leq i \leq \frac{n-5}{2},$
$p(z_1) = 4n + 4,$
$p(z_2) = 4n + 7.$



Radio labeling of WH10



Radio labeling of WH11

Figure 4

Examples of radio labeling(as define in Theorem 3.9) for n = 10 and n = 11 are shown in Figure 4.
Claim: The labeling **f** is a valid radio labeling i.e. the radio condition

$d(u, v) + |f(u) - f(v)| \geq 1 + diam(WH_n) \geq 6$
must holds for all distinct pairs of vertices of WH_n. We will discuss two cases for n.

When nis even:

Case 1: In this case we consider the pairs of vertices (v_i, v_j), (u_i, u_j), where $1 \leq i, j \leq n$. Consider the pair (v_i, v_j). As $d(v_i, v_j) \leq 2$ and $f(v_i) \in \{1, 5, 9, 15, 19, \ldots, 2n+3, 2n+7, 2n+11, \ldots, 4n-1\}$. The possible label difference for each pair will satisfy $|f(v_i) - f(v_j)| \geq 4$. So, the radiocondition becomes

$$d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 2 + 4, \text{ or } d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 6.$$

Hence, the radio condition is satisfied. Similarly we can check the radio condition for the pair of vertices (u_i, u_j).

Case 2: In this case we consider the pairs of vertices (z_1, z_2), (z_1, v_i), (z_1, u_i), (z_2, v_i) and (z_2, u_i), where $1 \leq i \leq n$. we will check the radio condition for (z_1, z_2), (z_1, v_i) and (z_1, u_i).

Subcase 2.1: Consider the pair (z_1, z_2). Since $d(z_1, z_2) = 3$, the radio condition becomes

$$d(z_1, z_2) + |f(z_1) - f(z_2)| = 3 + |4n + 4 - 4n - 7| = 6.$$

Subcase 2.2: Consider the pair (z_1, v_i), where $z_1 = v_i$. As $d(z_1, v_i) = 1$ for $1 \leq i \leq n$. Since $f(v_i) \in \{1, 5, 9, 15, 19, \ldots, 2n+3, 2n+7, 2n+11, \ldots, 4n-1\}$ and $f(z_1) = 4n + 4$. So, the radio condition becomes

$$d(z_1, v_i) + |f(z_1) - f(v_i)| \geq 1 + |4n+4-4n+1|, \text{ or } d(z_1, v_i) + |f(z_1) - f(v_i)| \geq 6.$$

Subcase 2.3: Consider the pair (z_1, u_i), where $z_1 = u_i$. As $d(z_1, u_i) \leq 4$ for $1 \leq i \leq n$. Since $f(u_i) \in \{2, 8, 12, 16, 21, 25, \ldots, 2n+13, 2n+17, \ldots, 4n+1\}$ and $f(z_1) = 4n + 4$. So, the radio condition becomes

$$d(z_1, u_i) + |f(z_1) - f(u_i)| \geq 4 + |4n+4-4n-1|, \text{ or } d(z_1, u_i) + |f(z_1) - f(u_i)| \geq 7,$$
$$\text{or } d(z_1, u_i) + |f(z_1) - f(u_i)| \geq 6.$$

Hence, the radio condition is satisfied in subcase 2.1, 2.2 and 2.3. Similarly we can check the radio condition for the pairs of vertices (z_2, v_i) and (z_2, u_i).

Case 3: Finally, consider (v_i, u_j), where $1 \leq i, j \leq n$. As $d(v_i, u_j) \leq 5$ we have $f(v_i) \in \{1, 5, 9, 15, 19, \ldots, 2n + 3, 2n + 7, 2n + 11, \ldots, 4n - 1\}$ and $f(u_i) \in \{2, 8, 12, 16, 21, 25, \ldots, 2n + 13, 2n + 17, \ldots, 4n + 1\}$. The labeldif- ferencefor each pair will satisfy $|f(v_i) - f(u_j)| \geq 1$. So, the radio conditionbecomes

$$d(v_i, u_j) + |f(v_i) - f(u_j)| \geq 5 + 1 = 6.$$

When n is odd:

Case 1: In this case we consider the pairs of vertices (v_i, v_j) and (u_i, u_j), where $1 \leq i, j \leq n$. Consider the pair (v_i, v_j), where $1 \leq i, j \leq n$. As $d(v_i, v_j) \leq 2$ and $f(v_i) \in \{1,5, 9, 15, 19, \ldots, 2n+5, 2n+9, 2n+13, \ldots, 4n-1\}$. The possible label difference for each pair will satisfy $|f(v_i)-f(v_j)| \geq 4$. So, the radio condition becomes

$$d(v_i, v_j) + |f(v_i) - f(v_j)| \geq 2 + 4 = 6.$$

Hence, the radio condition is satisfied in this case. Similarly we can check the radio condition for the pair of vertices (u_i, u_j) for $1 \leq i, j \leq n$.

Case 2: In this case we consider the pairs of vertices (z_1, z_2), (z_1, v_i), (z_1, u_i), (z_2, v_i) and (z_2, u_i), where $1 \leq i \leq n$. We will check the radiocondition for (z_1, z_2), (z_1, v_i) and (z_1, u_i).

Subcase 2.1: Consider the pair (z_1, z_2). Since $f(z_1) = 4n + 4$, $f(z_2) = 4n + 7$ and $d(z_1, z_2) = 3$. So, the radio condition becomes

$$d(z_1, z_2) + |f(z_1) - f(z_2)| = 3 + |4n + 4 - 4n - 7| = 6.$$

Subcase 2.2: Consider the pair (z_1, v_i), where $z_1 = v_i$. As $d(z_1, v_i) = 1$ for $1 \leq i \leq n$. Since $f(v_i) \in \{1,5, 9, 15, 19, \ldots, 2n+5, 2n+9, 2n+13, \ldots, 4n-1\}$ and $f(z_1) = 4n + 4$. So, the radio condition becomes

$$d(z_1, v_i)+|f(z_1)-f(v_i)| \geq 1+|4n+4-4n+1|, \text{ or } d(z_1, v_i)+|f(z_1)-f(v_i)| \geq 1+5, \text{ or } d(z_1, v_i) + |f(z_1) - f(v_i)| \geq 6.$$

Subcase 2.3: Consider the pair (z_1, u_i), where $z_1 = u_i$. As $d(z_1, u_i) \leq 4$ for $1 \leq i \leq n$. Since $f(u_i) \in \{2,8, 12, 16, 21, 25, \ldots, 2n+11, 2n+15, \ldots, 4n+1\}$ and $f(z_1) = 4n + 4$. So, the radio condition becomes

$$d(z_1, u_i)+|f(z_1)-f(u_i)| \geq 4+|4n+4-4n-1|, \text{ or } d(z_1, u_i)+|f(z_1)-f(u_i)| \geq 7, \text{ or } d(z_1, u_i) + |f(z_1) - f(u_i)| \geq 6.$$

Hence, the radio condition is satisfied in subcase 2.1, 2.2 and 2.3. Similarly we can check the radio condition for the pairs of vertices (z_2, v_i) and (z_2, u_i).

Case 3: Finally, consider (v_i, u_j) for $1 \leq i, j \leq n$. As $d(v_i, u_j) \leq 5$. We have $f(v_i) \in \{1,5, 9, 15, 19, \ldots, 2n + 5, 2n + 9, 2n + 13, \ldots, 4n - 1\}$ and $f(u_i) \in \{2,8, 12, 16, 21, 25, \ldots, 2n + 11, 2n + 15, \ldots, 4n + 1\}$. The possible difference of labels for each pair will satisfy $|f(v_i)-f(u_j)| \geq 1$. The radiocondition becomes

$$d(v_i, u_j) + |f(v_i) - f(u_j)| \geq 5 + 1, \text{ or } d(v_i, u_j) - |f(v_i) - f(u_j)| \geq 6.$$

Hence, the radio condition is satisfied in this case.

These three cases(for n is even and odd) establish the claim that f is a valid radio labeling of WH_n. *Thus,* $rn(WH_n) \leq span(f) = 4n + 7$.

Note: For n = 3, diameter of WH_3 is 3. It is easy to find that the radio number of WH_3 is 12. For $4 \leq n \leq 9$, the $rn(WH_n)$ cannot be found usingthe above procedure. Itseasy to see that for $3 \leq n \leq 9$, we have

Theorem 3.10. For $n \geq 10$ the radio number of Joint Wheel graph(WH_n)is

$$rn(WH_n) = 4n + 7.$$

Proof. Follows from Theorem 3.8 and Theorem 3.9.

Open problem: Investigate the $rn(FW^k_n)$ when the copies of Wheel graphhasdifferent number of vertices.

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