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ONE MODULO N GRACEFULNESS OF REGULAR BAMBOO TREE AND COCONUT TREE

V.Ramachandran1 C.Sekar2

1 Department of Mathematics, P.S.R Engineering College (Affiliated to Anna University Chennai), Sevalpatti,
Sivakasi, Tamil Nadu, India.
2 Department of Mathematics, Aditanar College of Arts and Science (Affiliated to MS University Tirunelveli),
Tiruchendur, Tamil Nadu, India.

Abstract

A function f is called a graceful labelling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function φ from the vertex set of G to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$ in such a way that (i) φ is 1 − 1 (ii) φ induces a bijection φ_ from the edge set of G to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$ where φ_(uv)=|φ(u) − φ(v)| . In this paper we prove that the every regular bamboo tree and coconut tree are one modulo N graceful for all positive integers N .

1. INTRODUCTION

S.W.Golomb [2] introduced graceful labelling. Odd gracefulness was introduced by. B.Gnanajothi [1] . C.Sekar [6] introduced one modulo three graceful labelling. V.Ramachandran and C.Sekar [4] introduced the concept of one modulo N graceful where N is any positive integer. In the case N = 2 , the labelling is odd graceful and in the case N = 1 the labelling is graceful. [6] Every regular bamboo tree is graceful. In this paper we establish the result for one modulo N graceful (N > 1) of the regular bamboo tree and also we prove that coconut tree is one modulo N graceful for all positive integers N . In order to prove the existing conjecture

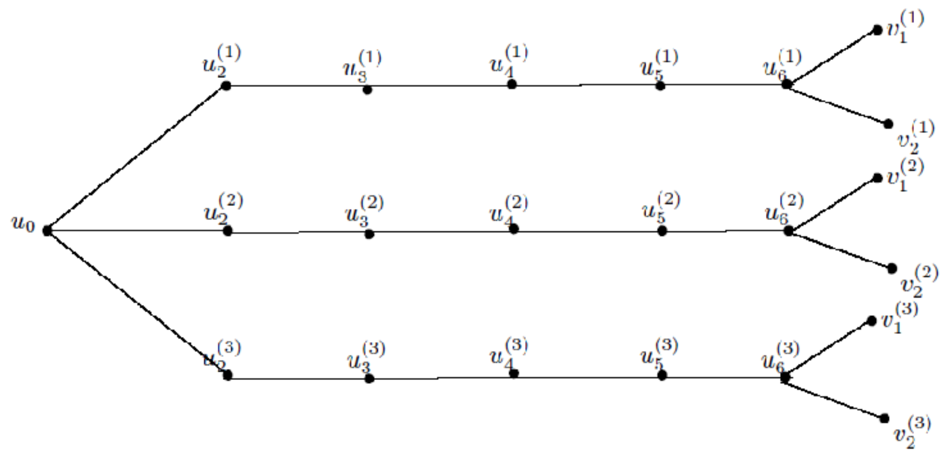
Problem 1. All trees are graceful?
Problem 2. All lobsters are graceful?

we take a diversion to prove one modulo N graceful of acyclic graphs. Sometimes the technique involved in one modulo N graceful labelling may yield a new approach to have graceful labelling of graphs. Our approach will motivate the scholars to do more research in this area.

2 Main Results Definition 2.1. A graph G with q edges is said to be one modulo N graceful (where N is a positive integer) if there is a function φ from the vertex set of G to {0, 1,N, (N + 1), 2N, (2N + 1), . . . ,N(q − 1),N(q − 1) + 1} in such a way that (i) φ is 1 − 1 (ii) φ induces a bijection φ_ from the edge set of G to {1,N + 1, 2N + 1, . . . ,N(q − 1) + 1} where φ_(uv)=|φ(u) − φ(v)| .

Definition 2.2. Consider k copies of paths Pn of length n−1 and stars Sm with m pendant vertices. Identify one of the two pendant vertices of the j th path with the centre of the j th star. Identify the other pendant vertex of each path with a single vertex u0 (u0 is not in any of the star and path).The graph obtained is a regular bamboo tree. Definition 2.3. A coconut Tree CT(m, n) is the graph obtained from the path Pn by appending m new pendent edges at an end vertex of Pn . Theorem 2.4. Every regular bamboo tree is one modulo N graceful for every positive integer N > 1.

Proof: Let $u_1^{(j)}, u_2^{(j)} \ldots, u_n^{(j)}$ be the vertices of the j th path where $u_1^{(j)}$ is identified with u_0 an $u_n^{(j)}$ is identified with $v_0^{(j)}$ which is the centre of the j th star. Let $v_1^{(j)}, v_2^{(j)} \ldots, u_n^{(j)}$ be the pendar vertices of the j th star. The bamboo tree has $k(n+m-1)+1$ vertices and $k(n+m-1)$ edges. Namin of the vertices is as shown in the figure.



Case (i) k is odd and n is odd
Define

$\phi(u_0) = \phi(u_1^{(j)}) = 0$

For $i = 2, 4, \ldots, n-1$

$\phi(u_i^{(j)}) = Nk(n+m-1) - (N-1) - N(j-1) - \frac{Nk(i-2)}{2} \quad \text{for } j = 1, 2, \ldots, k$

For $i = 3, 5, \ldots, n$

$$\phi(u_i^{(j)}) - \begin{cases} N(k+1) + N(j-1) + \frac{Nk(i-3)}{2} & \text{for } j - 1, 2, \ldots, \frac{(k-1)}{2} \\ \frac{N(k+1)}{2} + N(j - \frac{(k+1)}{2}) + \frac{Nk(i-3)}{2} & \text{for } j = \frac{(k+1)}{2}, \frac{(k+3)}{2}, \ldots, k \end{cases}$$

For $r - 1, 2, \ldots, m$

$\phi(v_r^{(j)}) - Nk(n+m-1) - (N-1) - \frac{Nk(n-1)}{2} - Nk(r-1) - N(j-1) \quad \text{for } j - 1, 2, \ldots, k$

From the definition of ϕ it is clear that

$\{\phi(u_0)\} \cup \{\phi(u_i^{(j)}),\ i = 2, 3, \ldots, n$ and $j = 1, 2, \ldots, k\} \cup \{\phi(v_r^{(j)}), r = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k\} = \{0\} \cup \{Nk(n+m-1) - N + 1, Nk(n+m-2) - N + 1, \ldots, Nk(\frac{n+2m+1}{2}) - N + 1, Nk(n+m-1) - 2N + 1, Nk(n+m-2) - 2N + 1, \ldots, Nk(\frac{n+2m+1}{2}) - 2N + 1, \ldots, Nk(n+m-2) + 1, Nk(n+m-3) + 1, \ldots, Nk(\frac{n+2m-1}{2}) + 1\} \cup \{N(k+1), N(2k+1), \ldots, Nk(\frac{n-1}{2}) + N, N(k+2), N(2k+2), \ldots, Nk(\frac{n-1}{2}) + 2N, \ldots, \frac{N}{2}(3k-1), \frac{N}{2}(5k-1), \ldots, \frac{N}{2}(nk-1), \frac{N}{2}(k+1), \frac{N}{2}(3k+1), \ldots, \frac{N}{2}(nk-k+1), \frac{N}{2}(k+3), \frac{N}{2}(3k+3), \ldots, \frac{N}{2}(nk-2k+3), \ldots, Nk, 2Nk, \ldots, \frac{Nk}{2}(n-1)\} \cup \{\frac{Nk}{2}(n+2m-1) - N + 1, \frac{Nk}{2}(n+2m-3) - N + 1, \ldots, \frac{Nk}{2}(n+1) - N + 1, \frac{Nk}{2}(n+2m-1) - 2N + 1, \frac{Nk}{2}(n+2m-3) - 2N + 1, \ldots, \frac{Nk}{2}(n+1) - 2N + 1, \ldots, \frac{Nk}{2}(n+2m-3) + 1, \frac{Nk}{2}(n+2m-5) + 1, \ldots, \frac{Nk}{2}(n-1) + 1\}$

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$.

We compute the edge labelling in the following sequence.

For $1 \leq j \leq k$

$\mid \phi(u_2^{(j)}) - \phi(u_0) \mid = Nk(n+m-1) - Nj + 1$

For $1 \leq r \leq m$ and $j = 1, 2, \ldots, \frac{k-1}{2}$

$\mid \phi(v_r^{(j)}) - \phi(u_n^{(j)}) \mid = Nk(m-r+1) - 2Nj + 1$

For $1 \leq r \leq m$ and $j = \frac{k+1}{2}, \frac{k+3}{2}, \ldots, k$

$\mid \phi(v_r^{(j)}) - \phi(u_n^{(j)}) \mid = Nk(m-r+2) - 2Nj + 1$

For $j = 2, 4, \ldots, n-1$ and $j = 1, 2, \ldots \frac{k-1}{2}$

$\mid \phi(u_i^{(j)}) - \phi(u_{i+1}^{(j)}) \mid = Nk(n+m-i) - 2Nj + 1$

For $j = 2, 4, \ldots, n-1$ and $j = \frac{k+1}{2}, \frac{k+3}{2}, \ldots, k$

$\mid \phi(u_i^{(j)}) - \phi(u_{i+1}^{(j)}) \mid = Nk(n+m-i+1) - 2Nj + 1$

For $j = 3, 5, \ldots, n-2$ and $j = 1, 2, \ldots \frac{k-1}{2}$

$\mid \phi(u_{i+1}^{(j)}) - \phi(u_i^{(j)}) \mid = Nk(n+m-i) - 2Nj + 1$

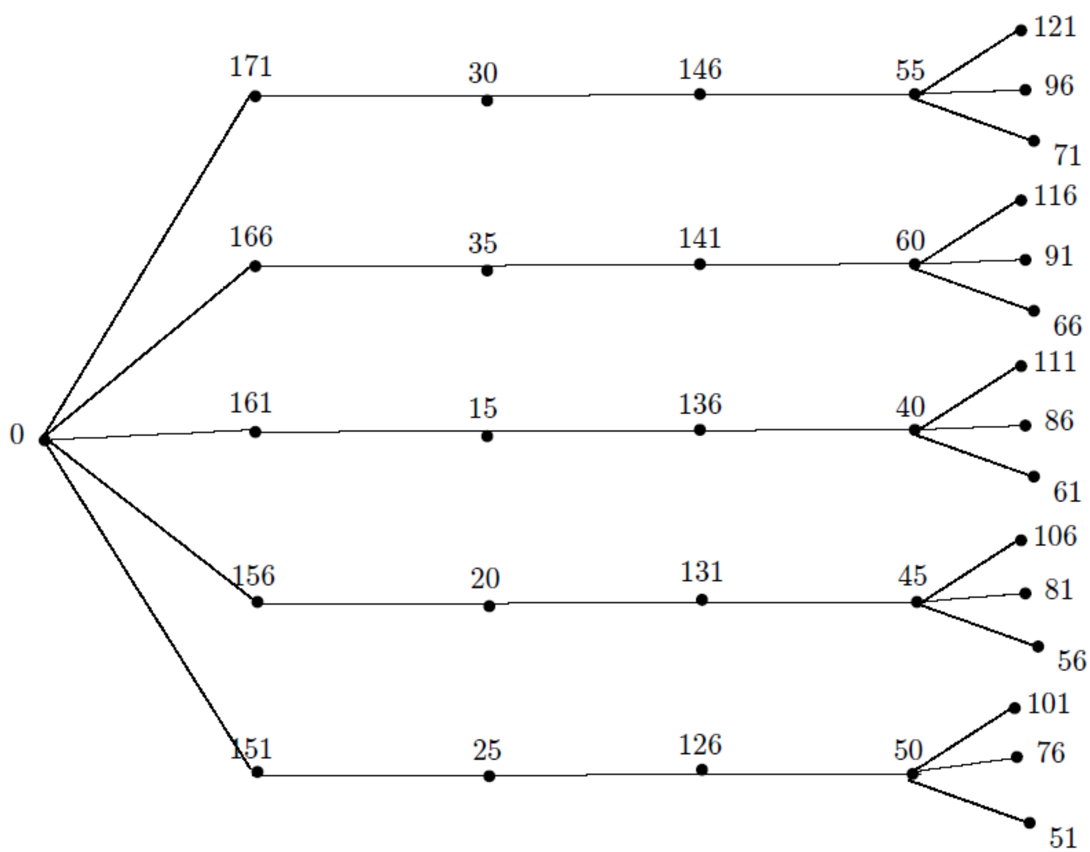
For $j = 3, 5, \ldots, n-2$ and $j = \frac{k+1}{2}, \frac{k+3}{2}, \ldots, k$

$\mid \phi(u_{i+1}^{(j)}) - \phi(u_i^{(j)}) \mid = Nk(n+m-i+1) - 2Nj + 1$

This shows that the edges have the distinct labels $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$.

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$. Hence the regular bamboo tree is one modulo N graceful.
Clearly ϕ defines a one modulo N graceful labelling of regular bamboo tree.

Example 2.5. One modulo 5 graceful labelling of regular bamboo tree. (k = 5 , n = 5 ,m = 3)



Case (ii) k is odd and n is even
Define

$\phi(u_0) = \phi(u_1^{(j)}) = 0$

For $i = 2, 4, \ldots, n$

$\phi(u_i^{(j)}) = Nk(n+m-1) - (N-1) - N(j-1) - \frac{Nk(i-2)}{2}$ for $j = 1, 2, \ldots, k$

For $i = 3, 5, \ldots, n-1$

$$\phi(u_i^{(j)}) = \begin{cases} N(k+1) + N(j-1) + \frac{Nk(i-3)}{2} & \text{for } j = 1, 2, \ldots, \frac{(k-1)}{2} \\ \frac{N(k+1)}{2} + N(j - \frac{(k+1)}{2}) + \frac{Nk(i-3)}{2} & \text{for } j = \frac{(k+1)}{2}, \frac{(k+3)}{2}, \ldots, k \end{cases}$$

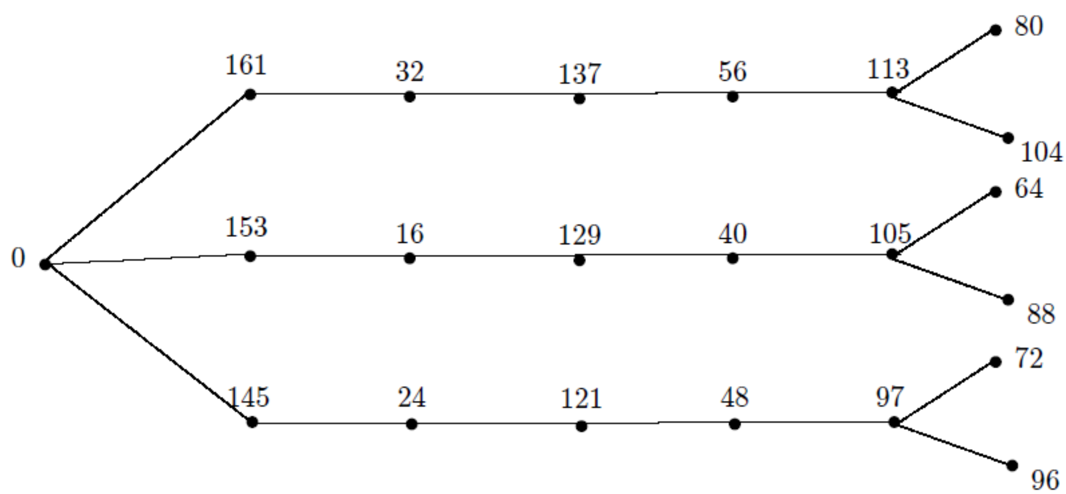
For $r = 1, 2, \ldots, m$

$$\phi(v_r^{(j)}) = \begin{cases} \frac{N(kn+2)}{2} + Nk(r-1) + N(j-1) & \text{for } j = 1, 2, \ldots, \frac{(k-1)}{2} \\ \frac{N(k(n-1)+1)}{2} + Nk(r-1) + N(j - \frac{(k+2)}{2}) & \text{for } j = \frac{(k+1)}{2}, \frac{(k+3)}{2}, \ldots, k \end{cases}$$

The proof is similar to the proof in case(i).

Clearly ϕ defines a one modulo N graceful labelling of regular bamboo tree.

Example 2.6. *One modulo* 8 *graceful labelling of regular bamboo tree.* $(k = 3, n = 6, m = 2)$



Case (iii) k is even and n is odd

Define

$\phi(v_r^{(1)}) = N(r-1) \quad \text{for } r = 1, 2, \ldots, m$

$\phi(u_0) = Nk(n+m-1) - (N-1) - \frac{N(n-1)}{2}$

$$\phi(u_i^{(1)}) = \begin{cases} N(m-1) + \frac{N(n-1)}{2} - \frac{N(i-2)}{2} & \text{for } i = 2, 4, \ldots, n-1 \\ Nk(m+n-1) + 1 - \frac{N(n-1)}{2} + \frac{N(i-3)}{2} & \text{for } i = 3, 5, \ldots, n \end{cases}$$

For $i = 3, 5, \ldots, n$

$\phi(u_i^{(j)}) = Nk(n+m-1) - (N-1) - N(j-2) - \frac{N(n-1+k)}{2} - \frac{N(k-1)(i-3)}{2} \quad \text{for } j = 2, 3, \ldots, k$

For $i = 2, 4, \ldots, n-1$

$$\phi(u_i^{(j)}) = \begin{cases} Nm + \frac{N(n-1+k)}{2} + N(j-2) + \frac{N(k-1)(i-2)}{2} & \text{for } j = 2, 3, \ldots, \frac{k}{2} \\ N(m-1) + \frac{N(n-1)}{2} + N + N(j - \frac{k}{2} - 1) + \frac{N(k-1)(i-2)}{2} & \text{for } j = \frac{k}{2}+1, \frac{k}{2}+2, \ldots, k \end{cases}$$

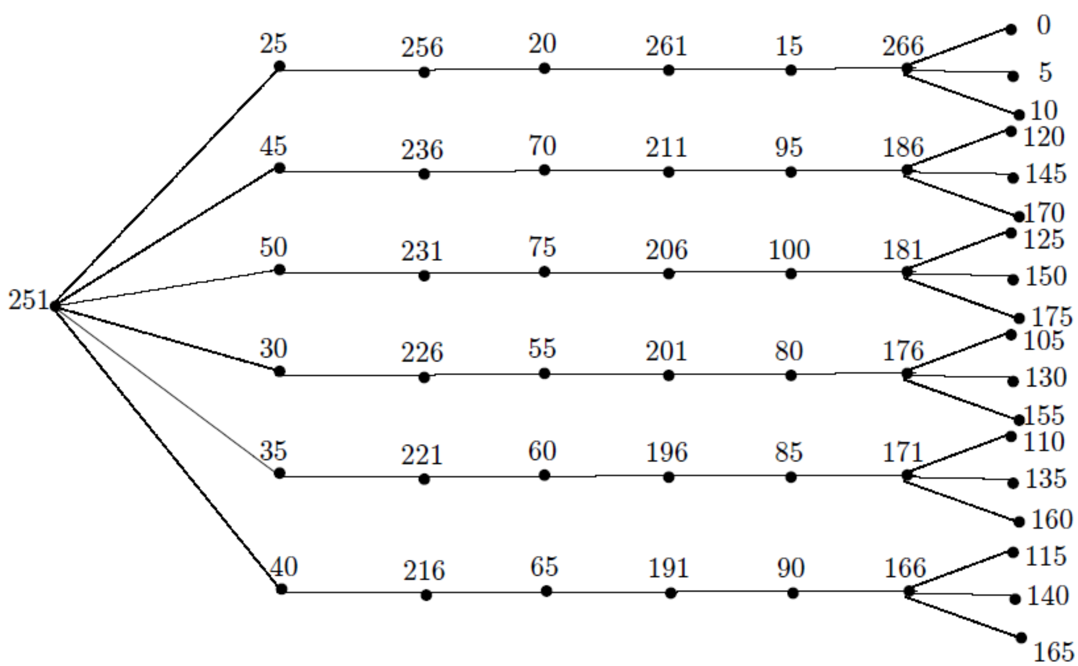
For $r = 1, 2, \ldots, m$

$$\phi(v_r^{(j)}) = \begin{cases} Nm + \frac{N(n-1+k)}{2} + N(j-2) + \frac{N(k-1)(n-3)}{2} + N(k-1) + N(k-1)(r-1) & \text{for } j = 2, 3, \ldots, \frac{k}{2} \\ Nm + \frac{N(n-1)}{2} + \frac{N(k-1)(n-3)}{2} + N(k-1) + N(j - \frac{k}{2} - 1) + N(k-1)(r-1) & \text{for } j = \frac{k}{2}+1, \ldots, k \end{cases}$$

The proof is similar to the proof in case(i).

Clearly φ defines a one modulo N graceful labelling of regular bamboo tree.

Example 2.7. One modulo 5 graceful labelling of regular bamboo tree. (k = 6 , n = 7 ,m = 3)



Case (iv) k is even and n is even

Define

$\phi(v_r^{(1)}) = N(r-1)$ for $r = 1, 2, \ldots, m$

$\phi(u_0) = N(m-1) - \frac{Nn}{2}$

$$\phi(u_i^{(1)}) = \begin{cases} Nk(m+n-1) - (N-1) - \frac{N(n-2)}{2} + \frac{N(i-2)}{2} & \text{for } i = 2, 4, \ldots, n \\ N(m-1) + \frac{N(n-2)}{2} - \frac{N(i-3)}{2} & \text{for } i = 3, 5, \ldots, n-1 \end{cases}$$

For $i = 2, 4, \ldots, n$

$\phi(u_i^{(j)}) = Nk(n+m-1) - (2N-1) - N(j-2) - \frac{N(n-2)}{2} - \frac{N(k-1)(i-2)}{2}$ for $j = 2, 3, \ldots, k$

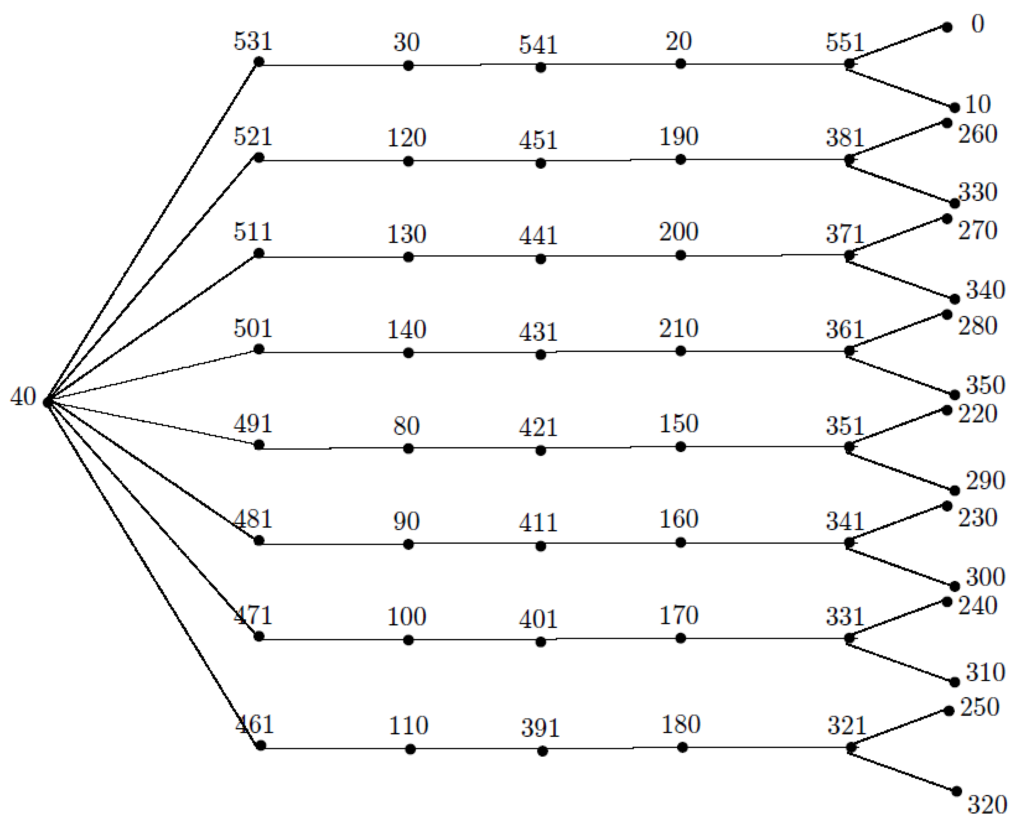
For $i = 3, 5, \ldots, n-1$

$$\phi(u_i^{(j)}) = \begin{cases} Nm + \frac{N(n+k)}{2} + N(j-2) + N(\frac{k}{2}-1) + \frac{N(k-1)(i-3)}{2} & \text{for } j = 2, 3, \ldots, \frac{k}{2} \\ Nm + \frac{N(n+K)}{2} + N(j-\frac{k}{2}-1) + \frac{N(k-1)(i-3)}{2} & \text{for } j = \frac{k}{2}+1, \frac{k}{2}+2, \ldots, k \end{cases}$$

For $r = 1, 2, \ldots, m$

$$\phi(v_r^{(j)}) = \begin{cases} Nm + \frac{N(n+k)}{2} + N(j-2) + \frac{N(k-1)(n-4)}{2} + N(\frac{k}{2}-2) + N(k-1)(r-1) & \text{for } j = 2, 3, \ldots, \frac{k}{2} \\ Nm + \frac{N(n+k)}{2} + \frac{N(k-1)(n-4)}{2} + N(k-2) + N(j-\frac{k}{2}-1) + N(k-1)(r-1) & \text{for } j = \frac{k}{2}+1, \ldots, k \end{cases}$$

The proof is similar to the proof in case(i).
Clearly φ defines a one modulo N graceful labelling of regular bamboo tree.
Example 2.8. One modulo 10 graceful labelling of regular bamboo tree. (k = 8 , n = 6 ,m = 2)



Theorem 2.9. *Coconut tree is one modulo N graceful for every positive integer N.*

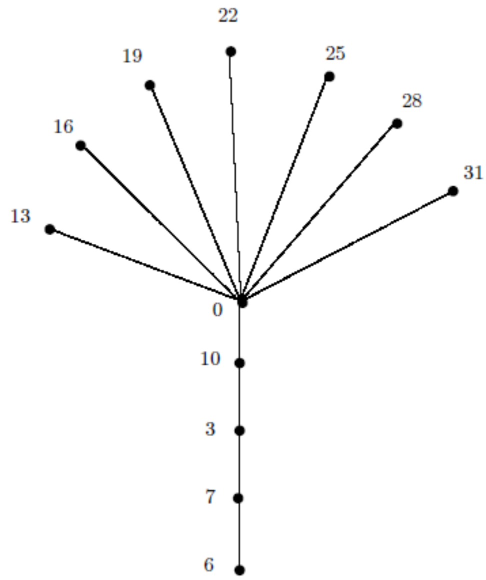
Proof: Let $u_1, u_2, \ldots, u_n$ be the vertices of a path P_n and $v_1, v_2, \ldots, v_m$ be the pendent vertices being adjacent with u_1 in the coconut tree G. Let c_i denote the edge $u_i u_{i+1}$ of P_n for $1 \le i \le n-1$ and $u_1 v_i$ for $1 \le i \le m$. The coconut tree G has $m+n$ vertices and $m+n-1$ edges.

Case (i) n is odd.
Let $n = 2k+1$, $k \ge 1$.
Define
$\phi(u_{2i-1}) = N(i-1)$ for $i = 1, 2, 3, \ldots, k+1$
$\phi(u_{2i}) = 2Nk - (N-1) - N(i-1)$ for $i = 1, 2, 3, \ldots, k$
$\phi(v_i) = 2Nk + 1 + N(i-1)$ for $i = 1, 2, 3, \ldots, m$
From the definition of ϕ it is clear that
$\{\phi(u_i), i = 1, 2, \ldots, n\} \cup \{\phi(v_i), i = 1, 2, \ldots, m\} = \{0, N, 2N, \ldots, Nk\} \cup \{N(2k-1)+1, N(2k-2)+1, \ldots, Nk+1\} \cup \{N(2k)+1, N(2k+1)+1, \ldots, N(2k+m-1)+1\}$

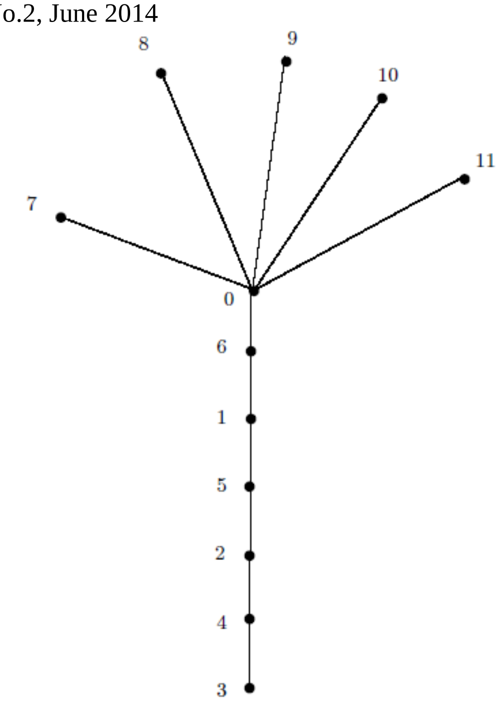
Thus it is clear that the vertices have distinct labels. Therefore φ is $1-1$.
We compute the edge labelling in the following sequence.
For $1 \le i \le m$
$|\phi(v_i) - \phi(u_1)| = N(2k+i-1)+1$
For $1 \le i \le k$
$|\phi(u_{2i}) - \phi(u_{2i-1})| = N(2k+1-2i)+1$
$|\phi(u_{2i}) - \phi(u_{2i+1})| = N(2k-2i)+1$
This shows that the edges have the distinct labels $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$.

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$.
Hence the coconut tree is one modulo N graceful.
Clearly ϕ defines a one modulo N graceful labelling of coconut tree.

Example 2.10. *One modulo 3 graceful labelling and graceful labelling of coconut tree*



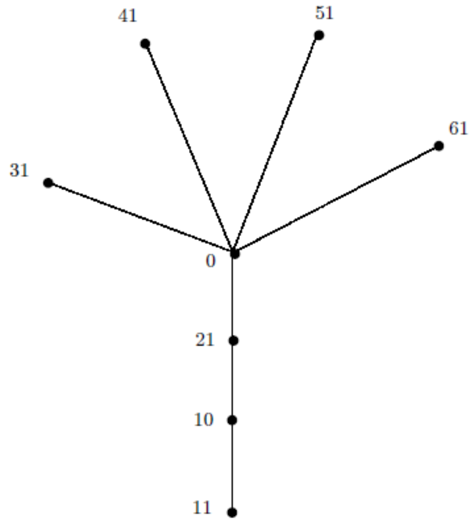
$n = 5$



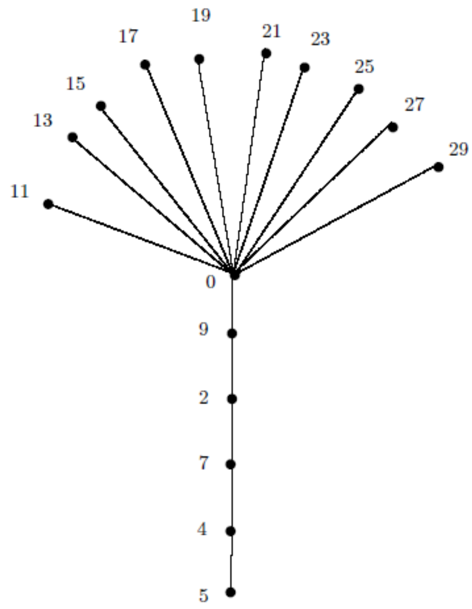
$n = 7$

Case (ii) n is even.
Let $n = 2k$.
Define
$\phi(u_{2i-1}) = N(i-1)$ for $i = 1, 2, 3, \ldots, k$
$\phi(u_{2i}) = 2Nk - (2N-1) - N(i-1)$ for $i = 1, 2, 3, \ldots, k$
$\phi(v_i) = 2Nk + 1 + N(i-1)$ for $i = 1, 2, 3, \ldots, m$
The proof is similar to the proof in case(i).
Clearly ϕ defines a one modulo N graceful labelling of coconut tree.

Example 2.11. *One modulo* 10 *graceful labelling and odd graceful labelling of coconut tree.*



$n = 4$



$n = 6$

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