AN APPLICATION OF G_d-METRIC SPACES AND METRIC DIMENSION OF GRAPHS

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Abstract

The idea of metric dimension in graph theory was introduced by P J Slater in [2]. It has been found applications in optimization, navigation, network theory, image processing, pattern recognition etc. Several other authors have studied metric dimension of various standard graphs. In this paper we introduce a real valued function called generalized metric $G_d : X \times X \times X \to R^+$ *where* $X = r(v/W) = \{(d(v,v_1), d(v,v_2),...,d(v,v_k)) / v \in V(G))\}$*, denoted* G_d *and is used to study metric dimension of graphs. It has been proved that metric dimension of any connected finite simple graph remains constant if* G_d *numbers of pendant edges are added to the non-basis vertices.*

Keywords

Resolving set, Basis, Metric dimension, Infinite Graphs, G_d *-metric.*

1. Introduction

Graph theory has been used to study the various concepts of navigation in an arbitrary space. A work place can be denoted as node in a graph, and edges denote the connections between places. The problem of *minimum machine (or Robots)* to be placed at certain nodes to trace each and every node exactly once is worth investigating. The problem can be explained using networks where places are interconnected in which, a navigating agent moves from one node to another in the network. The places or nodes of a network where we place the machines (robots) are called '*landmarks*'. The minimum number of machines required to locate each and every node of the network is termed as "*metric dimension*" and the set of all minimum possible number of landmarks constitute "*metric basis*".

A discrete metric like generalized metric [14] is defined on the Cartesian product $X \times X \times X$ of a nonempty set X into R^+ is used to expand the concept of metric dimension of the graph. The definition of a generalized metric space is given in 2.6. In this type of spaces a non-negative real number is assigned to every triplet of elements. Several other studies relevant to metric spaces are being extended to *G*-metric spaces. Different generalizations of the usual notion of a metric space were proposed by several mathematicians such as G¨ahler [17, 18] (called 2-metric spaces) and Dhage [15, 16] (called D-metric spaces) have pointed out that the results cited by G¨ahler are independent, rather than generalizations, of the corresponding results in metric spaces. Moreover, it was shown that Dhage's notion of D-metric space is flawed by errors and most of the results established by him and others are invalid. These facts are determined by Mustafa and Sims [14] to introduce a new concept in the area, called G-metric space.

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The concept of metric dimension was introduced by P J Slater in [2] and studied independently by Harary and Melter in [3]. Applications of this navigation of robots in networks are discussed in [4] and in chemistry, while applications to problems of pattern recognition and image processing, some of which involve the use of hierarchical structures are given in [5]. Besides Kuller et.al. provide a formula and a linear time algorithm for computing the metric dimension of a tree in [1]. On the other hand Chartrand et.al. in [7] characterize the graph with metric dimension 1, n -1 and n -2. See also in [8] the tight bound on the metric dimension of unicyclic graphs. Shanmukha and Sooryanarayana [9,10] compute the parameters for wheels, graphs constructed by joining wheels with paths, complete graphs etc. In 1960's a natural definition of the dimension of a graph stated by Paul Erdos and state some related problems and unsolved problems in [11]. Some other application including coin weighing problems and combinatorial search and optimization [12]. The metric dimension of the Cartesian products of graph has been studied by Peters-Fransen and Oellermann [13].

The metric dimension of various classes of graphs is computed in [3, 4, 5, 9, 10]. In [4, 5] the results of [3] are corrected and in [9, 10] the results of [5] are refined.

2. Preliminaries

The basic definitions and results required in subsequent section are given in this section.

2.1. Definition

A graph $G=(V,E)$ is an ordered pair consisting of a nonempty set $V=V(G)$ of elements called *vertices* and a set $E=E(G)$ of unordered pair of vertices called *edges*.

Two vertices $u,v\in V(G)$ are said to be *adjacent* if there is an edge $uv\in E(G)$ joining them. The edge $uv\in E(G)$ is also said to be *incident* to vertices u *and* v. The degree of a vertex v, denoted by $\deg(v)$ is the number of vertices in $V(G)$ adjacent to it.

An edge of a graph is said to be a *pendant edge* if it is incident with only one vertex of the graph.

A uv-path is a sequence of distinct vertices $u=v_o,v_1,...,v_n=v$ so that v_{i-1} is adjacent to v_i for all $i,\ 1\le i\le n$, such a path is said to be of length n. A uu-path of length n is a cycle denoted by C_n.

A graph is said to be *connected* if there is a path between every two vertices. A *complete graph* is a simple graph (a graph having no loops and parallel edges) in which each pair of distinct vertices is joined by an edge.

2.2. Definition

A graph G is *infinite* if the vertex set $V(G)$ is infinite. An infinite graph is *locally finite* if every vertex has finite degree. An infinite graph is *uniformly locally finite* if there exists a positive integer M such that the degree of each vertex is at most M. For example, the infinite path P_∞ is both locally finite and uniformly locally finite by taking $M=2$.

2.3. Definition

If G is a connected graph, the distance $d(u,v)$ between two vertices $u,v\in$ V(G) is the length of the shortest path between them. Let $W=\{w_1,w_2,...,w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v/W)$ of v respect to W is the k-tuple $(d(v,w_1), d(v,w_2),...,d(v,w_k))$. If distinct vertices of G have distinct representations (*co-ordinates*)

with respect to W, then W is called a resolving set or location set for G. A resolving set of minimum cardinality is called a basis for G and this cardinality is called the metric dimension or location number of G and is denoted by *dim*(G) or $\beta(G)$.

For each landmark, the *coordinate* of a node 'v' in G having the elements equal to the cardinality of the set W and i^{th} element of coordinate of 'v' equal to the length of the shortest path from the i^{th} landmark to the vertex 'v' in G.

For example, consider the graph G of figure 1. The set $W_1 = \{v_1, v_2\}$ is not a resolving set of G

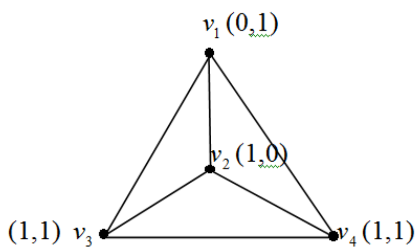

Figure 1.

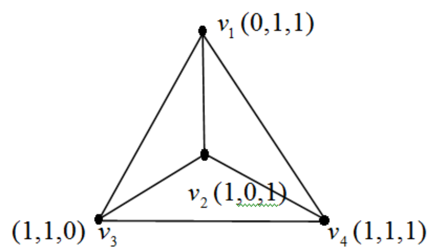

Figure 2.

Since $r(v_3/W_1) = (1,1) = r(v_4/W_1)$. Similarly, we can show that a set consisting of two distinct vertices will not give distinct coordinates for the vertices in G. On the other hand, $W_2 = \{v_1, v_2, v_3\}$ form a resolving set for G in figure 2, since the representation for the vertices in G with respect to W_2 are $r(v_1/W_2) = (0,1,1), r(v_2/W_2) = (1,0,1)$, $r(v_3/W_2) = (1,1,0)$, $r(v_4/W_2) = (1,1,1)$ and it is the minimum resolving set implying that $\dim(G) = 3$.

2.4. Remark

A graph can have more than one resolving set. For example consider the graph in figure 3. Here we obtained two resolving sets namely {a,b} and {a,c}.

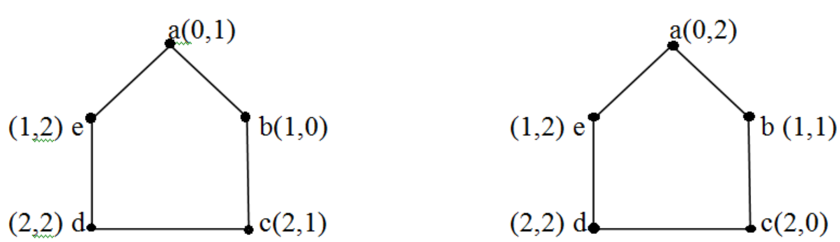

Figure 3. A graph with two resolving sets

2.5. Definition

Let X be a nonempty set. A G_d - *Metric or generalized metric* is a function from $X \times X \times X$ into R^+ having the following properties:

$G_d(x, y, z) = 0$ if $x = y = z$ for $x, y, z \in X$
$0 \le G_d(x, x, y)$ for all $x, y \in X$ with $x \ne y$
$G_d(x, x, y) \le G_d(x, y, z)$ for all $x, y, z \in X$ with $z \ne y$

$G_d(x, y, z) = G_d(x, z, y) = G_d(y, z, x) = \ldots$(symmetry in all the three variables) &
$G_d(x, y, z) \le G_d(x, a, a) + G_d(a, y, z)$ $\forall x, y, z, a \in X$ (Rectangle inequality)

2.6. Illustration

Let (X, d) be a metric space. Define $G_d : X \times X \times X \to R^+$ by
$G_d(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ is a G_d -metric satisfying the above five conditions.
Conversely if (X, G_d) is a G_d-metric space, it is easy to verify that (X, d_{G_d}) is a metric space where $d_{G_d}(x, y) = \frac{1}{2}(G_d(x, x, y) + G_d(x, y, y))$

For,

a) $d_{G_d}(x, y) = \frac{1}{2}(G_d(x, x, y) + G_d(x, y, y)) \ge 0$ by (ii)

b) $d_{G_d}(x, x) = \frac{1}{2}(G_d(x, x, x) + G_d(x, x, x)) = 0$ by (i)

c) $d_{G_d}(x, y) = \frac{1}{2}(G_d(x, x, y) + G_d(x, y, y)) = \frac{1}{2}(G_d(y, y, x) + G_d(y, x, x)) = d_{G_d}(y, x)$ by (iv)

d) $d_{G_d}(x, y) = \frac{1}{2}(G_d(x, x, y) + G_d(x, y, y))$

$$\le \frac{1}{2}\left[G_d(x, x, z) + G_d(x, z, z) + G_d(z, z, y) + G_d(z, y, y)\right]$$
$$\le d_{G_d}(x, z) + d_{G_d}(z, y)$$

Since $G_d(x, x, y) = G_d(y, x, x) \le G_d(y, z, z) + G_d(z, x, x)$
Similarly $G_d(x, y, y) \le G_d(x, z, z) + G_d(z, y, y)$ by (v)

Now we recall a few results already published in [23]

2.7. Theorem [7]

The metric dimension of graph G is 1 if and only if G is a path.

Figure 4. (black colored vertices shows the metric basis for P_∞)

2.8. Theorem [7]

If K_n is the complete graph with $n > 1$ then $\beta(K_n) = n - 1$.

2.9. Theorem [1]

If C_n is a cycle of length $n > 2$, then $\beta(C_n) = 2$.

2.9. Theorem [20]

If G is an infinite graph with finite metric dimension then it is uniformly locally finite.

The infinite graph $P_{2\infty}$ is uniformly locally finite with metric dimension equal to two.

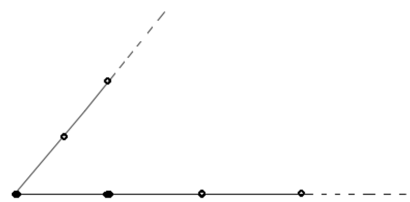

Figure 4.

The converse of the above theorem is not true. That is a uniformly locally finite graph need not have finite metric dimension. For example the infinite comp is uniformly locally finite but its metric dimension is infinite.

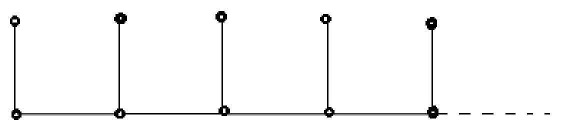

Figure 5.

3. Main Results

3.1. Theorem

The metric dimension of the graph obtained by adding 'n' pendant edges to each of the 'n' vertices in the complete graph K_n, $n > 2$ is same as that of K_n.

Proof: We have $\beta(K_n) = n-1$. Let $W = \{v_1, v_2, ..., v_n\} \setminus \{v_i\}$ for some i, $1 \le i \le n$ be a basis for K_n. Since every vertices K_n are adjacent to each other the coordinate of (n-1) vertices v_j, $j \ne i$ in W has (n-1) components at which j^{th} component takes the value '0' and the other components are 1's with respect to W. Now the vertex $v_i \notin W$ is adjacent to the vertices in W, its coordinate vector also has (n-1) components and that will be (1,1,…,1).

Suppose $m_1, m_2, ..., m_n$ are the pendant edges added correspondingly to the vertices $v_1, v_2, ..., v_n$ such that $m_j = (v_j, u_j), 1 \le j \le n$. Let the graph obtained in this way is denoted by $K = K_n + \sum_{j=1}^{n} m_j$. We know that the coordinate of v_j is (1,1,…,$0_{(j^{th}\text{ place})}$,1,…,1). So for some j, $d(v_j, u_j) = 1$ and since every vertex $v \in V(K_n)$ are adjacent to v_j, $d(v,u_j)=2$ for all those vertices $v \neq v_j$. Hence the coordinate of u_j will be $(2,2,\ldots,1_{(j^{th}\text{ place})},2,\ldots,2)$.

That is, the coordinate of u_1 is (1,2,…,2), u_2 is (2,1,…,2)…, u_n is (2,2,…,1) respectively. Thus the vertices in the graph K obtained by adding 'n' pendant edges to each of the vertices in K_n has distinct coordinates with respect to W. Therefore W itself is the basis for K_n and hence $\beta(K)=n-1$.

3.2. Illustration

Consider K_5 (Figure 6). Here five pendant edges $m_j=(v_j,u_j), 1\le j\le 5$ are added at each of the vertices $v_1,v_2,v_3,v_4,$ and v_5 respectively and shown that $\beta(K)=\beta\left(K_5+\sum_{j=1}^{5} m_j\right)=5-1=4$.

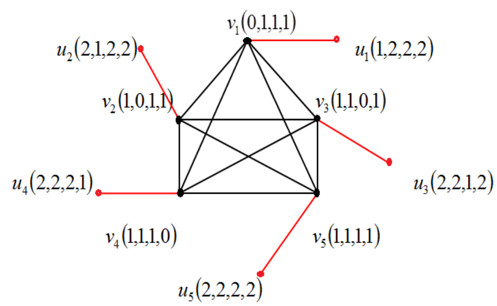

Figure 6.

The following corollary is about infinite graph with constant metric dimension.

3.3. Corollary

The above theorem holds for an infinite graph obtained by adding pendant edges $m_j=(v_j,u_j), 1\le j\le n$ successively at each u_j. Thus there exist infinite graphs with finite metric dimension.

The development of uniformly locally finite (ULF)[19] graphs is based on the adjacency operator A acting on the space of bounded sequences defined on the vertices. It has several applications in spectral theory. The following theorem gives a simple result on uniformly locally finite graph.

3.4. Theorem

The infinite graph $K=K_n+\sum_{j=1}^{\infty} m_j$ mentioned in theorem 3.1 is uniformly locally finite graph with finite metric dimension.

Proof: By theorem 3.1 $\beta(K)=\beta(K_n+\sum_{j=1}^{\infty} m_j)=n-1$ where $m_j=(v_j,u_j), 1\le j\le\infty$. Since every vertex is adjacent to each other in K_n, $d(v)=n-1$ for $v\in V(K_n)$ and the degree of the vertices u_j which is one of the end vertex in each of the edge added to K_n is 2. Now fix a positive integer $M=n-1$ where $n>2$. Then $d(v)\le M$ for all $v\in K$. Thus K is uniformly locally finite.

3.5. Theorem

Let G be connected graph with $\beta(G)=k$, $W=\{v_1,v_2,...,v_k\}$ be the basis and $X=r(v/W)=\{(d(v,v_1),d(v,v_2),...,d(v,v_k))/v\in V(G)\}$. Define generalized metric or G_d-metric $G_d : X\times X\times X\to R^+$ by

$$G_d(\vec{x},\vec{y},\vec{z})=\min_{\vec{x},\vec{y},\vec{z}\in X\subseteq R^k}\{d(\vec{x},\vec{y}),d(\vec{x},\vec{z}),d(\vec{y},\vec{z})\}$$

Where d is the 2-metric defined from $X\times X\to R^+$ by $d(\vec{x},\vec{y})=\sum_{i=1}^{k}|x_i-y_i|$.

If $G_d(\vec{x},\vec{y},\vec{z})=m$ then the metric dimension of the super graph $\tilde{G}$ obtained by adjoining at most m pendant edges to the vertices $v\notin W$ is same as that of G with respect to W. That is $\beta(\tilde{G})=\beta(G)$.

Proof: Let $W=\{v_1,v_2,v_3...,v_k\}$ be the basis for G. Then the coordinate space $r(v/W)=\{d(v,v_1),d(v,v_2),...,d(v,v_k)/v\in V(G)\}$. Since $\beta(G)=k$, the coordinate of each vertex in G contains 'k' components and they are distinct.

Let $G_d(\bar{x},\bar{y},\bar{z})=m$. Now we add m pendant edges are added to suitable vertices $v\notin W$. Suppose the first pendant edge e_1 is added at $v_j\notin W$ and $e_1=(v_j,v_{e_1})$. The coordinate of v_j is $(d(v_j,v_1),d(v_j,v_2),...,d(v_j,v_k))$ and it is distinct from the coordinate of other vertices in G.

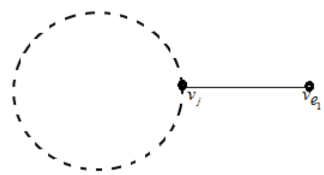

Figure 7.

Thus the coordinate of v_{e_1} will be $(d(v_j,v_1)+1,d(v_j,v_2)+1,...,d(v_j,v_k)+1)$ with respect to W and is different from all other coordinates of the vertices in G since $(d(v_j,v_1),d(v_j,v_2),...,d(v_j,v_k))$ is distinct from $(d(v_i,v_1),d(v_i,v_2),...,d(v_i,v_k))$, $i=1,2,...,n, i\neq j$. Hence $\beta(G+e_1)=k$. If the second pendant edge is added at v_{e_1} say $e_2=(v_{e_1},v_{e_2})$, then by the same argument as in the case of v_{e_1}, the coordinate of v_{e_2} will be $(d(v_{e_1},v_1)+1,d(v_{e_1},v_2)+1,...,d(v_{e_1},v_k)+1)$ and it is distinct from all other coordinates $(d(v_j,v_1),d(v_j,v_2),...,d(v_j,v_k))$ for $j=1,2,...,n, j=e_1$. Then obviously the coordinate of the new vertex is distinct from all other vertices since each component in the coordinate of v_{e_2} is increased by one. Thus $\beta(G+e_1+e_2)=k$.

Suppose the second pendant edge e_2 is added to $v_i, i\neq j$ in $G+e_1$ and $e_2=(v_i,v_{e_2})$. Here also the coordinate of v_{e_2} will be $(d(v_i,v_1)+1,d(v_i,v_2)+1,...,d(v_i,v_k)+1)$. Hence $\beta(G+e_1+e_2)=k$. Therefore the result is true for $m=1,2$. Assume that $\beta(G+e_1+e_2+....+e_{m-1})=k$ where $e_l=(v_j,v_{e_l})$ for $v_j\notin W, l=1,2,...,m-1$. If e_m is added at any v_{e_l} then each of the 'k' components in the coordinate of the vertex v_{e_m} is increased by one and hence it is distinct from other coordinates. If e_m is added to any vertex v in G not in W and not the end vertex of any of e_l, $l=1,2,...,m-1$, then the coordinate of v_{e_m} will be $(d(v,v_1)+1,d(v,v_2)+1,......d(v,v_k)+1)$ and distinct from all other coordinates of the vertices in the super graph $G+e_1+e_2+....+e_{m-1}$. Thus

$\beta(G+e_1+e_2+....+e_{m-1}+e_m)=k$. Hence the result is true for m. Thus the theorem is true for any integral value of $G_d(\vec{x},\vec{y},\vec{z})\in R^+$.

3.6 Example

Consider a 5- vertex Kite say H (Figure 8). There $X=\{(0,1),(1,0)(1,1),(2,1),(3,2)\}$ and $G_d(\bar{x},\bar{y},\bar{z})=Min\{d(\bar{x},\bar{y}),d(\bar{y},\bar{z}),d(\bar{x},\bar{z})\},\bar{x},\bar{y},\bar{z}\in X =2$, where $d(\bar{x},\bar{y})=\sum_{i=1}^{2}|x_i-y_i|$

Thus the minimum number of pendant edges that added to the Kite is 2. If these edges are added to those vertices which are not in W namely v_3 and v_4 with $\beta(H+e_1+e_2)=2$.

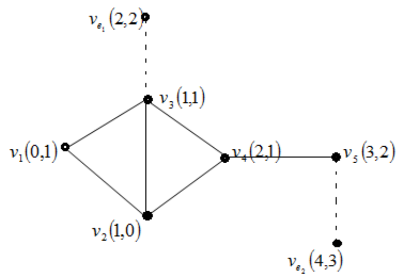

Figure 8.

3.7. Example

Consider C_4

$\beta(C_4)=2$ with respect to $W=\{v_1,v_2\}$ (Figure 9). Then $X=r(v/W)=\{(0,1),(1,0),(1,2),(2,1)\}$

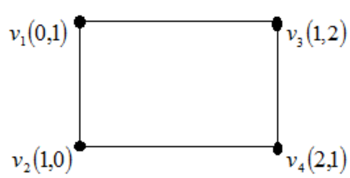

Figure 9.

By the definition of $G_d: X\times X\times X\to R^+$, we have

$$G_d(\bar{x},\bar{y},\bar{z})=Min\{d(\bar{x},\bar{y}),d(\bar{y},\bar{z}),d(\bar{x},\bar{z})\},\bar{x},\bar{y},\bar{z}\in X$$
$$=1, \quad \text{where } d(\bar{x},\bar{y})=\sum_{i=1}^{2}|x_i-y_i|$$

So one pendant edge is added to C_4. Suppose the pendant edge is added at $v_1\in W$ and $e_1=(v_1,v_{e_1})$. Then the coordinate of v_{e_1} is (1,2) with respect to W, but that is similar to the coordinate of v_3 (Figure 10). Therefore $\beta(C_4+e_1)\neq 2$ with respect to W.

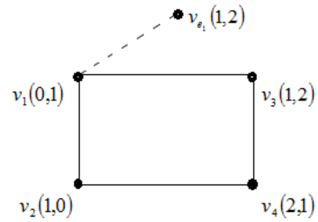

Figure 10.

Similarly if e_1 is added to $v_2 \in W$, the coordinate of v_{e_1} will be (2,1) and that is similar to the coordinate of v_4 (Figure 11). Thus e_1 must be added to any of v_3 or v_4. It will give a distinct representation for the coordinates of the vertices in $C_4 + e_1$ (Figure 12). That is e_1 must be added to the vertices not in W.

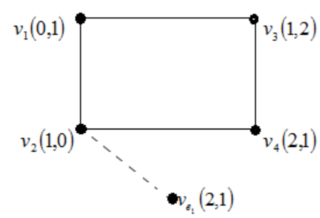

Figure 11.

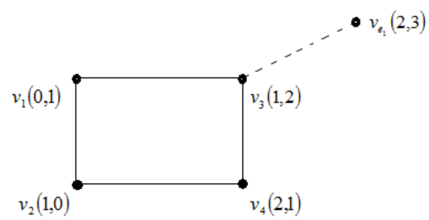

Figure 12.

Note: Since W is not unique, $\beta(C_4 + e_1) = 2$ with respect to another resolving set $W = \{v_{e_1}, v_2\}$ and $r(v/W) = \{(0,2),(1,1)(2,0),(2,2),(3,1)\}$ (Figure 13).

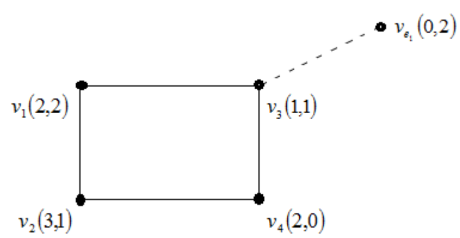

Figure 13.

4. Conclusion

This paper gives a measure that can be used in navigation space where the number of robots required to navigate a work place kept constant. Extension of navigation space will lead us to infinite graphs and its properties. With the help of G_d -metric and its properties we established general concepts and results.

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