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METRIC DIMENSION AND UNCERTAINTY OF TRAVERSING ROBOTS IN A NETWORK

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ABSTRACT

Metric dimension in graph theory has many applications in the real world. It has been applied to the optimization problems in complex networks, analyzing electrical networks; show the business relations, robotics, control of production processes etc. This paper studies the metric dimension of graphs with respect to contraction and its bijection between them. Also an algorithm to avoid the overlapping between the robots in a network is introduced.

KEYWORDS

Metric dimension, Contraction, Bijection, Overlapping, Cardinal number.

1.INTRODUCTION

If G is a connected graph with vertex set $V(G)=\{v_1,v_2,...,v_i,...,v_m\}, i=1,2,...,m$ where $m \geq n$ and let $W=\{v_1,v_2,...,v_j,...,v_n\}, j=1,2,...,n$ be an ordered set of vertices of G and v be a vertex of G. The coordinate of v with respect to W is the k-tuple $(d(v,v_1),d(v,v_2),...,d(v,v_n))$. If distinct vertices of G have distinct *co-ordinates*with respect to W , then W is called a resolving set or location set for G. A resolving set of minimum cardinality is called a basis for G and this cardinality is called the metric dimension or location number of G and is denoted by *dim*(G) or $\beta(G)$

The concepts of graph theory have been used to describe navigation in network. In a network, each place represented as nodes in a graph, and edges denote the connections between places. The places or nodes of a network where we place the machines (robots) are called landmarks. The minimum number of machines required to locate each and node of the network is termed as metric dimension and the set of all minimum possible number of landmarks constitute metric basis.

The concept of metric dimension was introduced by P. J.Slater in [2] and studied independently by Harary and Melter in [3]. Applications of navigation of robots in networks were discussed in [4]. Applications to problems of pattern recognition and image processing, which involving the use of hierarchical structures were done in [5]. Besides,Kuller et.al.[6]provideda formula and a linear time algorithm for computing the metric dimension of a tree in [1]. On the other hand, Chartrand et.al.in [7] characterized the graph with metric dimension 1, n -1 and n -2 and the tight bound on the metric dimension of unicyclic graphs[8]. Shanmukha and Sooryanarayana [9,10] computed the parameters for wheels, graphs constructed by joining wheels with paths, complete graphs etc. In 1960's Paul Erdos defined the dimension of a graph and stated some related problems and unsolved problems in [11]. The metric dimension of the Cartesian products of graph has been studied by Peters-Fransen and Oellermann [13].

The metric dimension of various classes of graphs was computed in [3, 4, 5, 9, 10].

2. PRELIMINARIES

This section summarizes basic definitions and results required in subsequent sections.

2.1. Definition

A graph $G=(V,E)$ is an ordered pair consisting of a nonempty set of vertices, $V=V(G)$ and a set of edges, $E=E(G)$. If the endpoints of an edge are equal then it is called a loop and edges having the same pair of endpoints arte called parallel edges. A graph having no loops and parallel edges is a simple graph. A *subgraph* of a graph G is a graph H such that $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$.

Two vertices are said to be adjacent if there is an edgejoining them. The number of vertices in $V(G)$ adjacent to v is the degree of v, denoted by $\deg(v)$.

A path is a sequence of distinct vertices $u=v_o, v_1, ..., v_n = v$ so that v_{i-1} is adjacent to v_i for all i, $1\le i\le n$, such a path is said to be of length n and if $u=v$ then it becomes a cycle of length n denoted by C_n.

A connected graph is the one in which there is a path between every two vertices. If each pair of vertices is adjacent then it is called complete graph.

A connected acyclic graph is called tree. A spanning subgraph of G is a subgraph with vertex set $V(G)$ and *spanning tree* is a spanning subgraph that is a tree.

Let $e=(u,v)$ be an edge in G. The contraction of edge e is the replacement of u and v with a single vertex and the edges other than e incident with this single vertex are those edges that were incident with u or v. The resulting graph is denoted by $G.e$ and $|E(G.e)| = |E(G)| - 1$.

2.2. Definition

Coordinate of a vertex v_i is represented by an n-tuple $(m_{i1}, m_{i2}, ..., m_{in}), i=1,2,...,m.$ In particular for the vertex v_1 the coordinate is $(m_{11}, m_{12}, ..., m_{1n})$ where $m_{11} = d(v_1, v_1), m_{12} = d(v_1, v_2)$ and so on.

2.3. Definition

Cardinal number of a basis element is denoted by $Ca(v_j), j=1,2,...,n, v_j \in W$ and is defines as the number of vertices of G identified by v_j with respect to $d(v_i, v_j) = l, i=1,2,...,m.$

For example, consider the graph given in Figure1 that has metric dimension two with respect to the basis $W=\{a,b\}$.

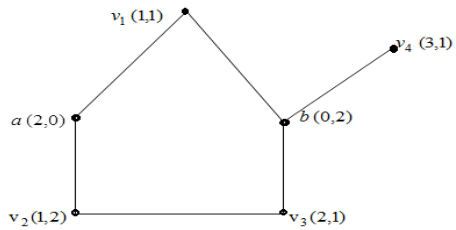


Figure 1

In the figure $Ca(a) = 2$ and $Ca(b) = 3$ with respect to $d(v_i, v_j) = 1$.

2.4. Definition

Two vertices u and v in W are said to be *overlapping* with each other if $Ca(u) = Ca(v)$ with respect to $d(v_i, u) = d(v_i, v) = l.$

In Figure 1 a and b are overlapping with respect to $d(v_1, a) = d(v_1, b) = 1.$

2.5. Robotic Assignment

Let $W = \{v_1, v_2, ..., v_j, ..., v_n\}, j = 1,2,...,n$ be the basis and $V(G) = \{v_1, v_2, ..., v_i, ..., v_m\}, i = 1,2,...,m$ be the vertex set of $G.$ For a vertex $v_i \in V(G)$, the coordinate is $(m_{i1}, m_{i2}, ..., m_{in}), i = 1,2,...,m.$ We can assign the basis element (Robot) v_j if $Min(m_{i1}, m_{i2}, ..., m_{in}) = m_{ij}$ for a particular $j.$

Example 1: In Figure 1 the coordinate of vertex v_2 is (1, 2) with respect to the basis $W = \{a, b\}.$ Here, $Min(1,2) = 1$, therefore basis element a is assigned to v_2.

Note 1: Suppose there is an alternative minima for j and $j+1.$ Then we can assign v_j to v_i if $Ca(v_j) < Ca(v_{j+1}).$

In Figure 1 the coordinate of v_1 is (1,1). Therefore $d(a, v_1) = d(b, v_1) = 1$ and here we can assign a to v_1 since $Ca(a) < Ca(b)$ where $Ca(a) = 2$ and $Ca(b) = 3$ with respect to $l = 1.$

Note 2: If $Ca(a) = Ca(b)$ then arbitrarily any one can assigned to a vertex v_i with respect to $d(a, v_i) = d(b, v_i) = 1.$ In Figure 2 for the vertex v , $Min(1,1) = 1$ and $Ca(a) = Ca(b) = 1$ with respect to $l = 1.$ So assign a or b to v .

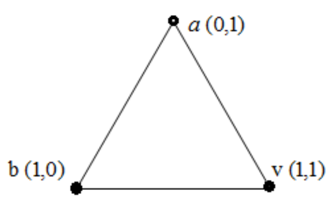


Figure 2

2.6. Robotic Assignment Subgraph

After making the Robotic Assignment we may obtain a spanning subgraph called Robotic Assignment spanning subgraph(RASS). Figure 3 represents the RASS for the graph given in figure 1.

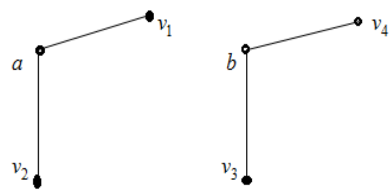


Figure 3

Now we recall a few results already published in [15,16]

2.7. Theorem[7]

The metric dimension of graph G is 1 if and only if G is a path.

Figure 4

2.8. Theorem [7]

The metric dimension of a complete graph with n vertices is $n-1$ where $n>1$.

2.9. Theorem [14]

Let $\tau(G)$ denote the number of spanning trees of a graph G . If $e\in E(G)$ is not a loop then $\tau(G)=\tau(G-e)+\tau(G.e)$.

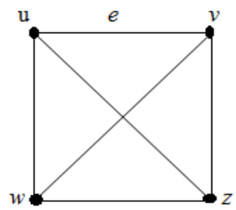
3. MAIN RESULTS

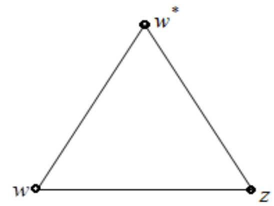
3.1. Theorem

Consider the graph K_n with n vertices $v_1,v_2,...,v_n$, then $\beta(K_n.e)=n-2$. where $K_n.e$ denote the contracted graph having no loops and parallel edges.

Proof: We have $\beta(K_n)=n-1$. Let $e=uv$ be an edge in K_n where u and v are adjacent to every other $n-2$ vertices. Consider the contracted graph $K_n.e$. Since every vertex in $K_n.e$ are adjacent to the other $n-2$ vertices. Here the edge $e\notin E(K_n.e)$ and the vertices u and v are replaced by a single vertex say w^*. Evidently the remaining $n-2$ vertices must adjacent to w^*

since both u and v are adjacent to those vertices. Then the simple graph $K_n.e$ contains exactly $n-1$ vertices and it should be K_{n-1}. Clearly $\beta(K_{n-1}) = n-1-1 = n-2$. Hence the proof.
Figure 5 represents K_4 and its contraction (simple graph) with the edge e.





K_4

Simple graph of $K_4.e$

Figure 5

For large scale computations, network models consist of several nodes and can place uniquely a minimum number of Robots to identify them. But in the case of optimization we cannot assign two machines (Robots) to the same node. The following algorithm explains that how we can avoid the overlapping between the machines.

3.2. Theorem (Algorithm)

Let G be a simple connected graph with $V(G) = \{v_1, v_2, ..., v_i, ..., v_m\}, i = 1,2,...,m.$ and $W = \{a_1, a_2, ..., a_j, ..., a_n\}, j = 1,2,...,n$ be the basis of G where $a_j = v_i$ for some i and $j = 1,2,...,n$ & $n \le m$ we can find any two adjacent vertices in G since it is connected.

The following steps yields a Robotic assignment subgraph for G.

Step 1: If $d(a_1, v_i) < d(a_j, v_i)$ for $j = 2,3,...,n$ then assign a_1 to v_i and vice versa for $i = 1,2,...,m$ and $v_i \notin W$.

Step 2: Let a_1 is adjacent to any of the vertex v_i for some i that is not in W and $d(a_1, v_i) = 1$. Consider all other a_j such that $d(a_j, v_i) = 1$.

Step 3: If $Ca(a_1) < Ca(a_j)$ for $j = 2,3,...,n$ then assign a_1 to v_i and add the edge (a_1, v_i) or in other words v_i is identified by the basis element a_1.

Step 4: Now take $d(a_j, v_i) = 2$ for some j and v_i is ma vertex which is not previously identified by any of the basis element. Consider all $a_k, k = 1,2,..., j-1, j+1,...,n$ at which $d(a_k, v_i) = 2$. If $Ca(a_j) < Ca(a_k)$ for $k = 1,2,..., j-1, j+1,...,n$ then assign a_j to v_i and join the path from a_j to v_i.

Step 5: Continue the process till all the vertices in G is identified by the any of the basis element in W.

Step 6: Joining an edge or path in this way we may obtain a connected or disconnected graph called Robotic assignment subgraph of G obviously it is spanning subgraph of G.

It can be easily verified that the graph in Figure 3 represents the Robotic assignment subgraph of the graph in Figure 1.

Note: If the Robotic assignment subgraph is not connected then adjoin the edge or path between any two vertices which are not in W to make it a connected spanning subgraph. This is possible since the graph is connected. Hence we obtain a spanning tree for G. The following figure represents the Robotic spanning tree S of the graph in Figure 1.

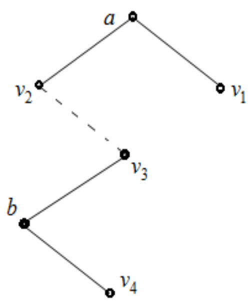


Figure 6

3.3. Adjacency matrix of S

It is interesting to know about the paths between basis elements and vertices of graphs. We know that the each entry in adjacency matrix $[S]_{n \times n}$ gives the path of length one between any two vertices in the graph, entry in $[S]^2$ gives the path of length two between any two matrices and so on. We are concentrating on the different path from basis elements to the vertices in the graph that are not in the basis. If we consider a sub matrix $[S]_{n \times n}$ of order $m \times n, m \leq n$ then each entry in that matrix gives path from Robots to nodes in the network. One can easily show that the diagonal of the matrices $[S]^2$ and MM^T are equal where M is the incidence matrix of S.

3.4. Example

Suppose the following graph represents the Robotic spanning tree S of some graph G and its adjacency matrix and incidence matrix are given in Figure 7.

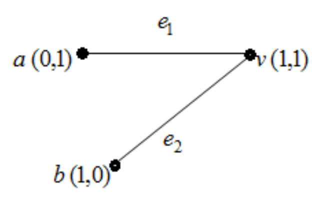


Figure 7

It is shown that the diagonals of $[S]^2$ and MM^T are same. That is we can easily find the route of length two from a robot to itself.

$$[S]=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\; M=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}$$

$$[S]^2=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$MM^T=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 2\end{bmatrix}$$

Since S is a spanning tree the number of different spanning trees of S, $\tau(S)=1$. The following theorem shows that there is a one to one correspondence between $E(S)$ and $\tau(S.\mathrm{e})$ with respect to a contraction. By means of contraction actually we reduce the cardinal number of each basis element. So contraction has great importance in the routing of complex networks.

3.5. Theorem

If S is the Robotic spanning tree of a graph G and $S.e$ is the contraction with respect to nonloop e then there exists a bijection between the edge set of S and $\tau(S.\mathrm{e})$.

Proof: Assume that no two basis elements in S are adjacent. For each edge $e \in E(S)$, the contraction $S.\mathrm{e}$ must be a tree since S itself a tree. Now $S.\mathrm{e}$ contains $\mathrm{E(S)\text{-}1}$ edges and $\mathrm{V(S)\text{-}1}$ vertices. If S contains m edges then corresponding to m edges we get m trees with respect to a contraction $S.\mathrm{e}$. Since each tree of $S.\mathrm{e}$ arises for exactly on edge from S and $S.\mathrm{e}$ itself a tree means $\tau(S.\mathrm{e})=1$. That means each contracted edge gives one and only one spanning tree of $S.\mathrm{e}$. Thus there is a bijection between $\mathrm{E(S)}$ and $\tau(S.\mathrm{e})$.

3.6. Example

Consider the Robotic spanning tree in the Figure 6. The figure given below shows the correspondence stated in the above theorem.

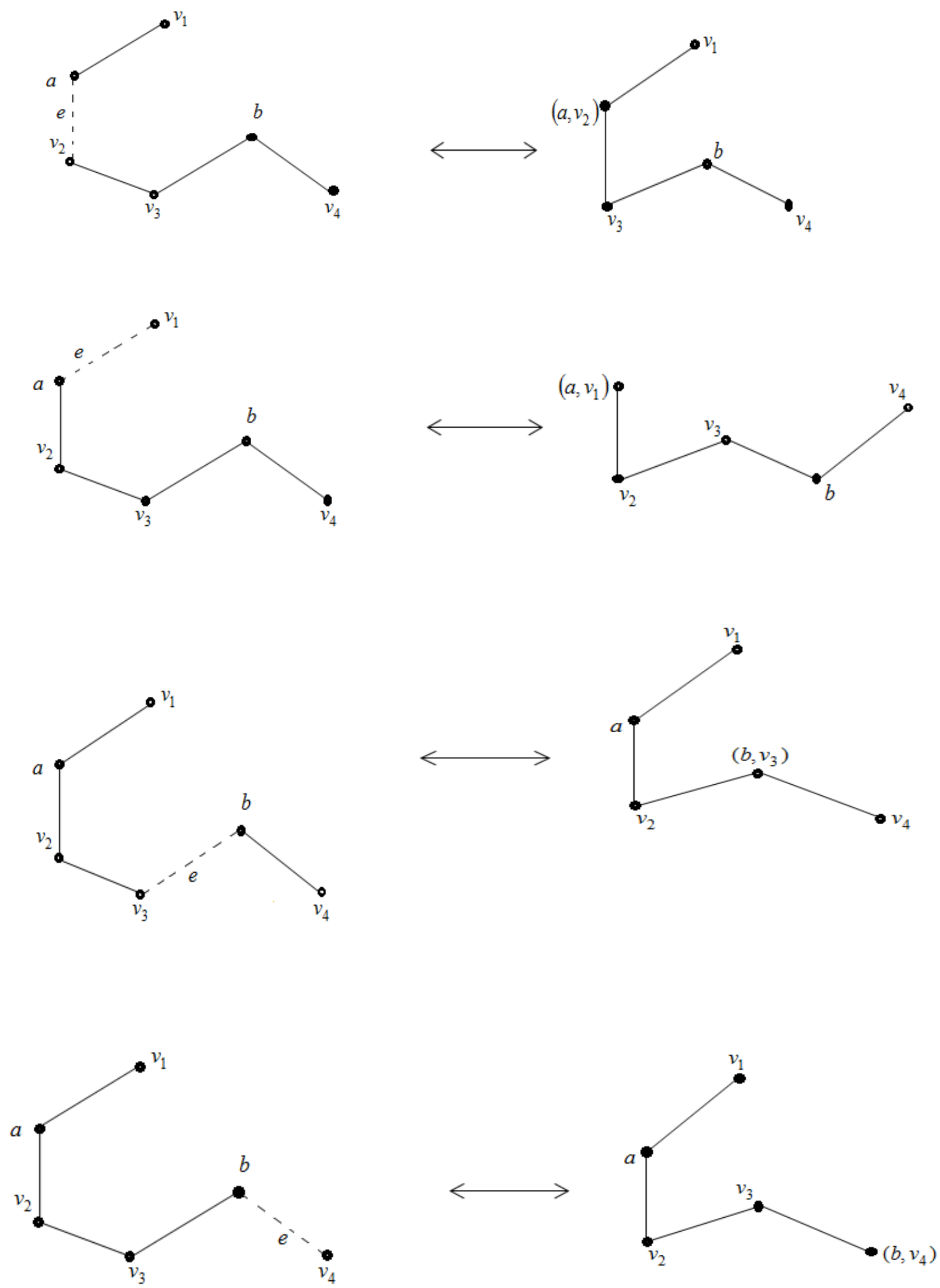


Figure 8

4. CONCLUSION

In this paper we studied the metric dimension of graphs with respect to contraction and its bijection between them. The algorithm proposed is used to avoid the overlapping between the robots in a network. This is applied in Robotic Assignment spanning subgraph in a complex network and provides an optimization through an algorithm in which the routing of basis elements helps in solving some complicated networks.

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